

Homological stability for oriented configuration spaces

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Abbreviate to $C_n(M)$ and $C_n^+(M)$ since we will keep X fixed throughout.

Stabilisation maps

$$C_n(M) \xrightarrow{s} C_{n+1}(M)$$

- assume $M =$ interior of manifold-with-boundary \bar{M}
- choose boundary-component of \bar{M} : $\partial_0 \bar{M}$
- s : push configuration away from $\partial_0 \bar{M}$ slightly, add a new point b_0 in the newly-vacated neighbourhood of $\partial_0 \bar{M}$ to the configuration.
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Oriented case: $C_n^+(M) \longrightarrow C_{n+1}^+(M)$

2 possible conventions for induced orientations; say

$$\begin{aligned} +s: \begin{bmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{bmatrix} &\mapsto \begin{bmatrix} p'_1 & \cdots & p'_n & b_0 \\ x_1 & \cdots & x_n & x_0 \end{bmatrix} \\ -s: \begin{bmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{bmatrix} &\mapsto \begin{bmatrix} p'_1 & \cdots & p'_{n-1} & b_0 & p'_n \\ x_1 & \cdots & x_{n-1} & x_0 & x_n \end{bmatrix} \end{aligned}$$

Homological stability

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- Rewrite this in terms of n : we want a function $k(n)$ so that

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- In our case k will always be linear, and we can talk about the *slope* of the stability range.

Brief history

	manifold M	space X	slope
Randal-Williams 10/11	[*]	path-conn.	$1/2$

[*] connected, dimension ≥ 2 , interior of a manifold-with-boundary

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Church 11	[**]	pt	1	with \mathbb{Q} -coefficients

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[**] connected, orientable, (finite-type)

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Answering the same question for $H_* C_n^+(M)$:

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NB: These proofs both involve calculations very specific to their respective situations, so do not generalise naturally.

Main theorem

In this talk we will outline the proof of homological stability for $C_n^+(M, X)$, with:

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$$M = \mathbb{R}^\infty \text{ and } X = pt \rightsquigarrow A_n$$

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More generally, taking $M = S =$ interior of a compact connected surface with boundary and $X = pt$ gives H.S. for $A\beta_n^S$.

Taking $X = BG$, we get H.S. for $A\beta_n^S \wr G$.

Main theorem

We will start by outlining the proof of the *unordered* version of this theorem:

Theorem (Randal-Williams)

$C_n(M) \xrightarrow{s} C_{n+1}(M)$ is

- an isomorphism on H_* for $* \leq n/2 - 1$
- a surjection on H_* for $* \leq n/2$.

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- a surjection on H_* for $* \leq n/2$.

This can be rephrased as

$$H_* R_n(M) = 0 \quad \text{for } * \leq n/2$$

where $R_n(M)$ is the **mapping cone** (homotopy cofibre) of $C_n(M) \xrightarrow{s} C_{n+1}(M)$.

Outline of proof: 1

Strategy — construct a map with target $R_n(M)$, and prove it is

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Outline of proof: 2: surjectivity on H_*

Factorise s into

$$C_{n-1}(M) \xrightarrow{p} C_{n-1}(M \setminus pt) \xrightarrow{f} C_n(M)$$

p — push configuration away from $\partial_0 \bar{M}$, remove b_0 from M .

f — fill in the missing point in the manifold, add this new point to the configuration.

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This is a factorisation of $\tilde{s}_{(12)}$.

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Using excision, can identify its **homotopy cofibre** (up to H_* -isom.) with that of the $\dim(M)$ -fold suspension of

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\tilde{f} : Involved spectral sequence argument, which I won't go into now...

Deferred to the end, if time.

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'Ladder trick':

The maps of cofibrations

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Now we will turn to the *oriented* version of the theorem:

Theorem (P.)

$C_n^+(M) \xrightarrow{s} C_{n+1}^+(M)$ is

- an isomorphism on H_* for $* \leq (n-2)/3 - 1$
- a surjection on H_* for $* \leq (n-2)/3$.

Main theorem – the oriented version

Now we will turn to the *oriented* version of the theorem:

Theorem (P.)

$C_n^+(M) \xrightarrow{s} C_{n+1}^+(M)$ is

- an isomorphism on H_* for $* \leq (n-2)/3 - 1$
- a surjection on H_* for $* \leq (n-2)/3$.

This can be rephrased as

$$H_* R_n^+(M) = 0 \quad \text{for } * \leq (n-2)/3$$

where $R_n^+(M)$ is the **mapping cone** (homotopy cofibre) of $C_n^+(M) \xrightarrow{s} C_{n+1}^+(M)$.

Outline of proof: 4: the oriented version

Note: We defined $s: \begin{bmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{bmatrix} \mapsto \begin{bmatrix} p'_1 & \cdots & p'_n & b_0 \\ x_1 & \cdots & x_n & x_0 \end{bmatrix}.$

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So in order for f and s to commute, we need to define

$$f: \begin{bmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{bmatrix} \mapsto \begin{bmatrix} * & p_1 & \cdots & p_n \\ x_0 & x_1 & \cdots & x_n \end{bmatrix}$$

(* = the missing point in $M \setminus pt$).

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(* = the missing point in $M \setminus pt$).

Upshot: the composition

$$C_{n-1}^+(M) \xrightarrow{p} C_{n-1}^+(M \setminus pt) \xrightarrow{f} C_n^+(M)$$

is a factorisation of $\boxed{(-1)^{n-1}s}$.

Outline of proof: 4: the oriented version

Now f and s commute exactly, and p and s commute up to the (12) -homotopy as before, so we get induced maps \tilde{f} , \tilde{p} on mapping cones, which factorise $\tilde{s}_{(12)}$:

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- \tilde{p} and \tilde{f} are surjective on H_* for $* \leq (n-2)/3$.

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Problems:

- ① The homotopy (12) does *not* split, so $\tilde{s}_{(12)}$ is not zero on H_* .
- ② The ‘ladder trick’ depends on knowing (in advance) that s induces injections on H_* ($\forall *$). In the oriented case this is **false**.

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- Show that each \tilde{f} , \tilde{p} is surjective on H_* for $* \leq (n-2)/3$. *This is where the 1/3 stability slope becomes necessary.*
- The other two homotopies, (123) and 1, split, so we have...

Solution to 2

- $\tilde{s}_{(132)}$ is **surjective** on H_* for $* \leq (n-2)/3$,
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Lemma

*The **existence** of a split homotopy (e.g. $\tilde{s}_{(123)}$ or \tilde{s}_1) filling the square implies that any homotopy (e.g. $\tilde{s}_{(132)}$) filling the square factorises as below.*

$$R_{n-2}^+(M) \longrightarrow \Sigma C_{n-2}^+(M) \dashrightarrow C_{n+1}^+(M) \longrightarrow R_n^+(M)$$

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Thanks for your attention!

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Appendix

Sketch of the spectral sequence argument deferred earlier.

Appendix

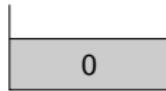
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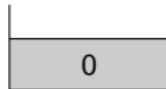
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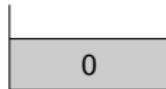
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