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Kontsevich characteristic classes à la Watanabe

reading seminar

Oxford

9 June 2020

following T. Watanabe

"Some exotic nontrivial elements of the rational
homotopy groups of $\text{Diff}(S^+)$ "

(penultimate talk)

Plan

1

- Recap of §4 so far : construction of $\pi^\Gamma: E^\Gamma \longrightarrow B_\Gamma$

from a trivalent graph Γ

- Proof that π^Γ is bundle-bordant to a bundle over S^k

- Addendum for §2 (formula for $\hat{Z}_k(\pi)$ when the base of π is not S^k)

- Recap of §3 : the "Z-graph counting formula" for $Z_k(\pi, \dots)$

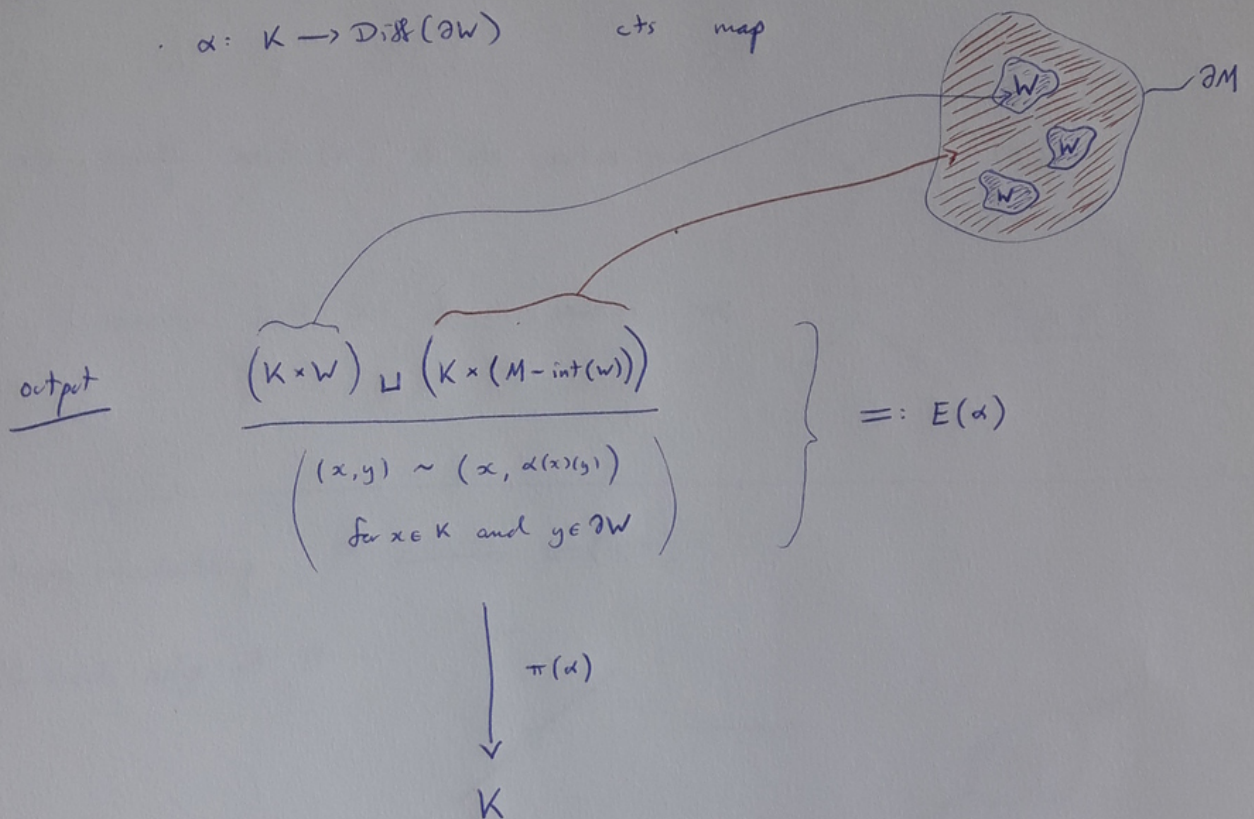
- "Key Lemma" $\rightarrow$ applying the Z-graph counting formula to $\pi^\Gamma: E^\Gamma \longrightarrow B_\Gamma$

Construction of $\pi^1: E^1 \rightarrow B_1$

2

General idea: (for constructing smooth M -bundles over K with trivial diskwise ∂)

input . $W \hookrightarrow \text{int}(M)$ codim-0 submanifold
 $\alpha: K \rightarrow \text{Disk}(\partial W)$ cts map



Input

Γ trivalent graph with $\text{edges} \longleftrightarrow \{1 \dots 3k\}$
 $\text{vertices} \longleftrightarrow \{1 \dots 2k\}$

& choose an orientation of each edge s.t. $\nexists$ source / sink



$\Rightarrow$ exactly half (k) of its vertices are



Type I

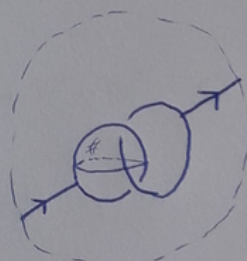
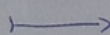
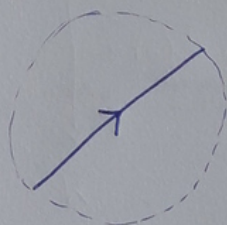
exactly half (k) of its vertices are



Type II

① Choose embedding $\Gamma \hookrightarrow \text{int}(D^4)$

② For each edge of Γ :



$\uparrow$
Hopf link $S^1 \cup S^2 \subseteq D^4$

$\leadsto \coprod_k \text{Type I} \cup \coprod_k \text{Type II} \hookrightarrow \text{int}(D^4)$

③ Take a small regular neighbourhood

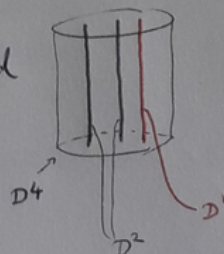
$\leadsto \coprod_k V_I \cup \coprod_k V_{II} \hookrightarrow \text{int}(D^4)$

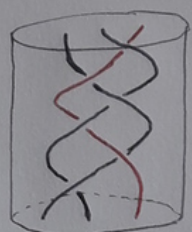
D^4
 $\cup$ two 1-handles
 $\cup$ one 2-handle

D^4
 $\cup$ one 1-handle
 $\cup$ two 2-handles

④ Watanabe defines $\alpha_I: S^0 \longrightarrow \text{Diff}(\partial V_I)$
 $\alpha_{II}: S^1 \longrightarrow \text{Diff}(\partial V_{II})$

$$\partial V_I \cong \partial \left(D^4 - (\text{two } 2\text{-handles and one } 1\text{-handle}) \right)$$

$$\cong \partial \left(\text{complement of } \underline{\text{trivial}} \text{ braid} \right)$$


$$\cong \partial \left(\text{complement of } \underline{\text{Borronean}} \text{ braid} \right)$$


↑ mapping cylinder of a diffeomorphism.

& similar idea for ∂V_{II} , using Borronean braid $D^3 \cup D^2 \cup D^2 \subseteq D^5$

This determines

$$\alpha_n: (S^0)^k \times (S^1)^k \longrightarrow \text{Diff}(\partial V_I)^k \times \text{Diff}(\partial V_{II})^k$$

$$\Downarrow$$

$$\text{Diff} \left(\partial \left(\coprod_k V_I \cup \coprod_k V_{II} \right) \right)$$

⑤ Applying the previous general construction, we get

$$E(\alpha_n) =: E^n$$

$$\downarrow \pi(\alpha_n) =: \pi^n$$

$$(S^0)^k \times (S^1)^k =: B_n$$

a D^4 -bundle with trivial ∂^{fb} .

Upshot:

- Γ labelled trivalent graph
- edge orientations without sources/sinks



(D^4, ∂) -bundle

E^π

$\downarrow \pi^\pi$

$$B_\pi = (S^0)^k \times (S^1)^k$$

Theorem (Watanabe) Assume that Γ is connected.

(a) π^π is bordant to a (D^4, ∂) -bundle over S^k

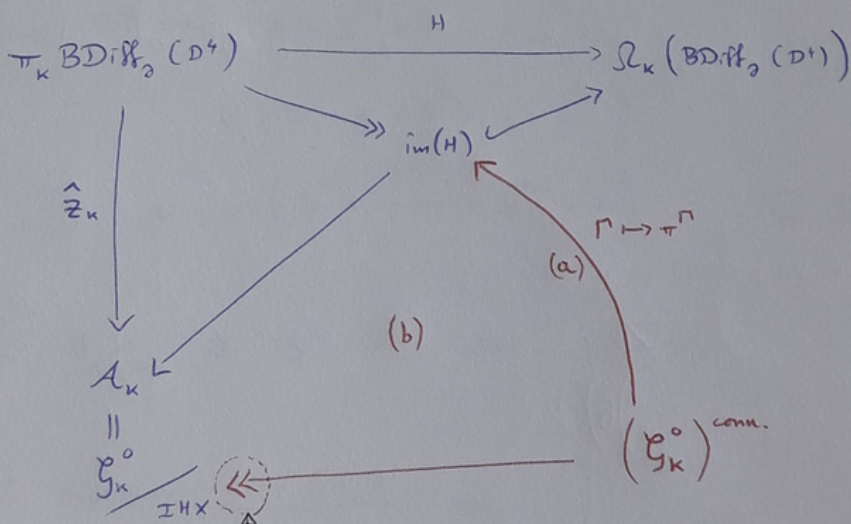
$\hookrightarrow$ hence $\hat{Z}_k(\pi^\pi) \in \mathcal{A}_k$ is defined!

(b) $\hat{Z}_k(\pi^\pi) = [\Gamma]$.

Coro : $\hat{Z}_k : \pi_k \text{BDiff}_2(D^4) \xrightarrow{\quad} \mathcal{A}_k$ is surjective

$\parallel$
 $\pi_{k-1} \text{Diff}_2(D^4)$

proof



every trivalent graph is
IHX-equivalent to a connected one.

□.

Proof of (b) — next week.

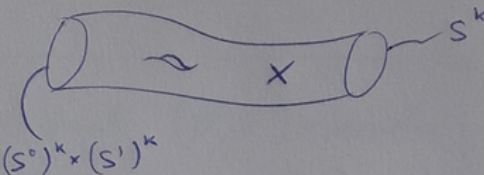
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Proof of (a) :

Under the correspondence

$$\left\{ (D^4, \partial) \text{-bundles over } B \right\} \longleftrightarrow \left\{ \text{maps } B \longrightarrow B\text{Diff}_2(D^4) \right\}$$

This is equivalent to the statements:

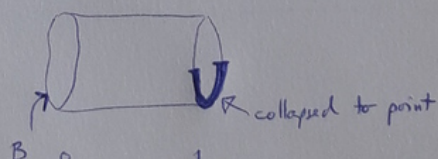
(i) $\exists$ bordism 

(ii) the classifying map of $\pi^1 : (S^0)^k \times (S^1)^k \longrightarrow B\text{Diff}_2(D^4)$ extends to X

For (i) : if $B = S^{a_1} \times \dots \times S^{a_r}$ is a product of spheres

$$\begin{aligned} \text{define } B^* &:= \left(* \times S^{a_2} \times \dots \times S^{a_r} \right) \\ &\cup \left(S^{a_1} \times * \times S^{a_3} \times \dots \times S^{a_r} \right) \\ &\cup \dots \cup \left(S^{a_1} \times \dots \times S^{a_{r-1}} \times * \right) \subseteq B. \end{aligned}$$

then $B/B^* \cong S^k$ where $k = \sum_i a_i$.

and moreover $\frac{B \times [0,1]}{B^* \times \{1\}} =$ 

is (a manifold and) a bordism from B to S^k .

For (ii).

To show that $\pi^n: B = (S^0)^k \times (S^1)^k \longrightarrow \text{BDiff}_2(D^4)$

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extends to $X = \frac{B \times [0,1]}{B^* \times \{1\}}$,

(*) { we need to show: π^n is homotopic to a map that sends $B^* \subseteq B$ to $* \in \text{BDiff}_2(D^4)$

We will prove:

$\pi^n|_{B[i]}$ is a trivial (D^4, ∂) -bundle

} (*)

where $B[i] = S^0 \times \dots \times S^0 \times S^1 \times \dots \times S^1 \subseteq B$

↖ with the i -th factor replaced by *

How this finishes the proof:

(*) $\Rightarrow$ the class map of π^n is nullhomotopic when restricted to $B[i]$

[extra technicalities] $\Rightarrow$ these nullhomotopies are compatible on $B[i] \cap B[j]$

$\searrow \Rightarrow \pi^n$ is nullhomotopic when restricted to $B^* = \bigcup_{i=1}^{2k} B[i]$

[$B^* \hookrightarrow B$ cofibration] $\Rightarrow$ this may be extended to a homotopy satisfying (*).

It remains to prove that $\pi^{\Gamma}|_{\text{BL}[i]}$ is a trivial (D^4, ∂) -bundle

(*) 8

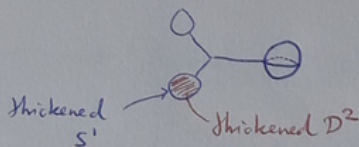
IDEA : $\text{Tub}\left(\begin{array}{c} \bigcirc \\ \bigcirc \end{array}\right) \quad \text{Tub}\left(\begin{array}{c} \bigcirc \\ \bigcirc \end{array}\right)$

Consider $V = V_{\text{I}} \approx V_{\text{II}} \subseteq \text{int}(D^4)$

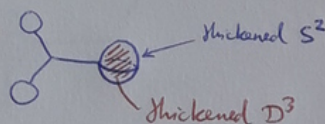
$\alpha = \alpha_{\text{I}} \approx \alpha_{\text{II}} : S^a \longrightarrow \text{Diff}(\partial V) \quad (a = 0 \text{ or } 1)$

Fact (1) Let $\hat{V}$ = obtained from V by

either (a) attaching a 2-handle to cancel
one of its 1-handles



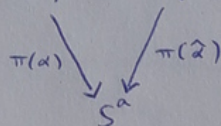
or (b) attaching a 3-handle to cancel
one of its 2-handles



$\subseteq \text{int}(D^4)$

Then $\exists \hat{\alpha} : S^a \longrightarrow \text{Diff}(\partial \hat{V})$

such that $E(\alpha) \cong E(\hat{\alpha})$



Fact (2) Moreover, $E(\hat{\alpha})$ is a trivial (D^4, ∂) -bundle.

Because $\alpha : S^a \longrightarrow \text{Diff}(\partial V)$ was constructed from a Borroneau braid.
Cancelling one handle to obtain $\hat{\alpha} : S^a \longrightarrow \text{Diff}(\partial \hat{V})$ corresponds to
removing a strand of the Borroneau braid
 $\leadsto$ trivial braid.

Proof of (*) :

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$\pi^r|_{B[i]}$ = (D^+, ∂) -bundle associated to

$$V_1 \cup \dots \cup V_{i-1} \cup V_{i+1} \cup \dots \cup V_{2k} \subseteq S_n^+(D^+).$$

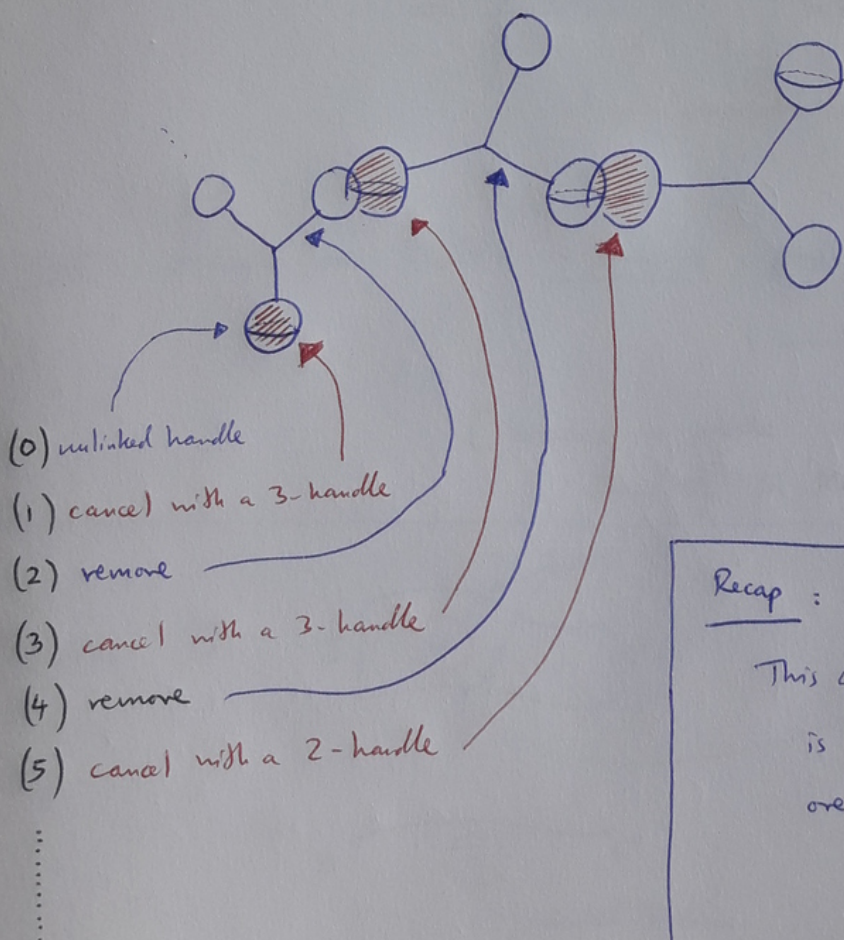
→ one of the V_j has an "unlinked" handle

Fact (1) → (D^+, ∂) -bundle unchanged if we replace $V_j \rightsquigarrow \hat{V}_j$.

Fact (2) → (D^+, ∂) -bundle unchanged if we remove $\hat{V}_j$.

Γ connected graph $\Rightarrow$ we may repeat until all V_j are removed.

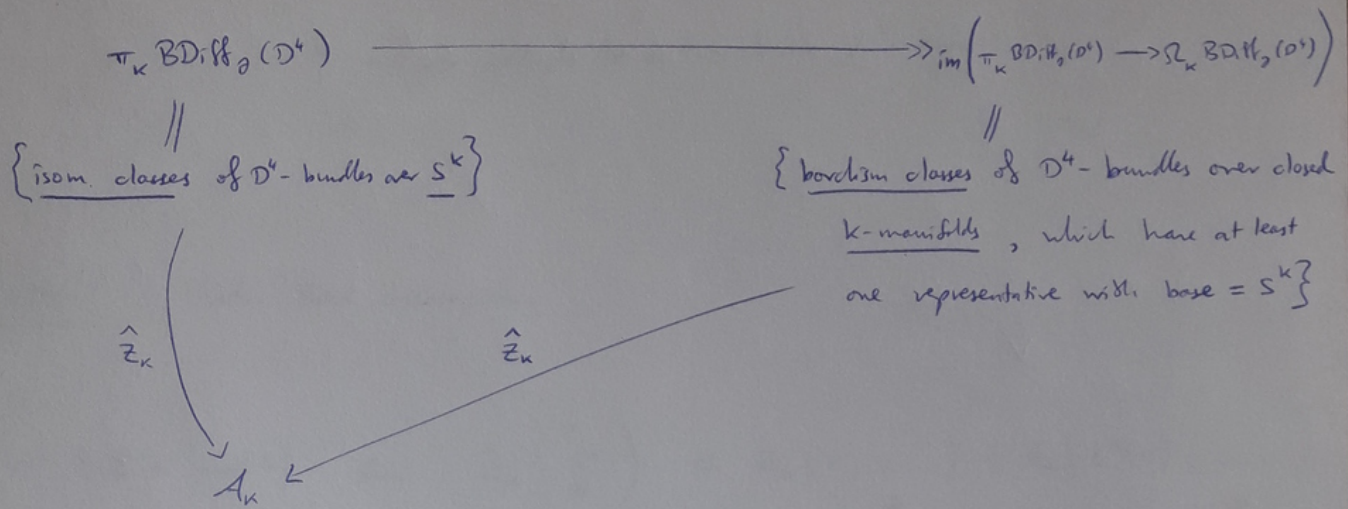
→ trivial (D^+, ∂) -bundle. $\square$



Recap :

This completes the proof that π^r is bordant to a (D^+, ∂) -bundle over S^k .

Recall that we defined:



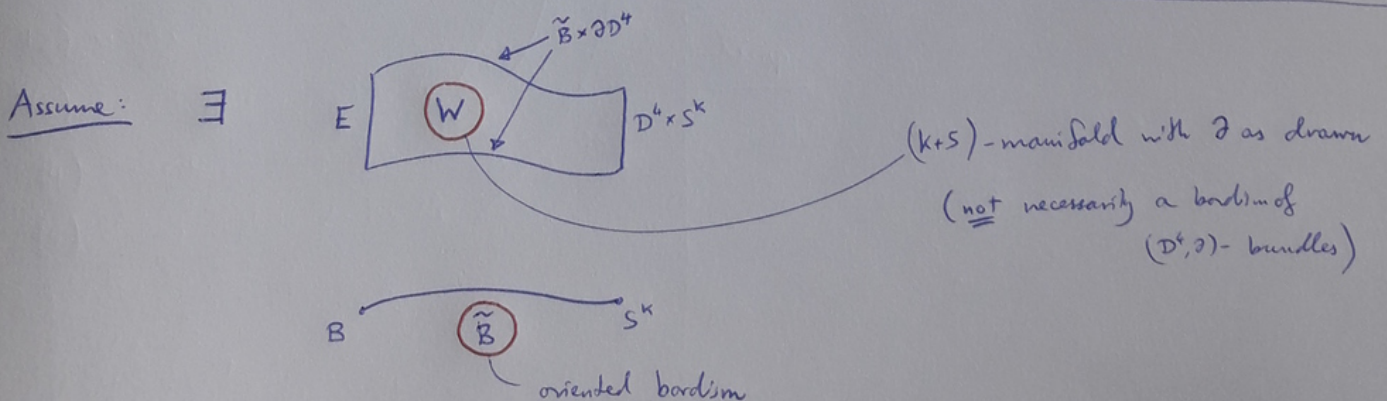
Hence
$$\hat{Z}_k \begin{pmatrix} E \\ \downarrow \pi \\ B \end{pmatrix} = \hat{Z}_k \begin{pmatrix} E_0 \\ \downarrow \pi_0 \\ S^k \end{pmatrix} \quad (\text{any choice of } \pi_0 \sim \pi)$$

$$= Z_k(\pi_0, \dots) - \alpha_k(\dots)$$

$\uparrow$
 adm. propagator for π_0

Want: formula for $\hat{Z}_k(\pi)$ in terms of propagator for π
 (instead of propagator for π_0)

[because in practice we understand π much better than π_0 in the examples we constructed]



& assume that $\begin{matrix} T^*E \\ \downarrow \\ E \end{matrix}$ is trivialisable (+ choose trivialisation)

III

→ extends to a trivialisable $\mathbb{R}^4$ -bundle $\begin{matrix} \varepsilon^4(W) \\ \downarrow \\ W \end{matrix}$

Proposition. Under these assumptions,

$$\text{if } k \not\equiv 3 \pmod{4} \text{ then } \hat{Z}_k \left(\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix} \right) = Z_k(\pi, \dots) - d_k(\varepsilon^4(W), \dots)$$

$\uparrow$ adm. propagators for π $\uparrow$ same chains on $S\varepsilon^4(W)$

$$\text{if } k \equiv 3 \pmod{4} \text{ then } \hat{Z}_k \left(\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix} \right) = \text{as above}$$

+ an extra term constructed using Pontryagin classes.