Manifold calculus & exotic Pontrjagin classes

Topology advanced class

7 Dec 2020

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1 Manifold calculus

<u>Recall</u> (Goodwillie - Weiss)

(functors in M)

$$\begin{tikzcd}
 & \mathrm{Emb}(M,N) \arrow[dl] \arrow[d] \arrow[dr, bend left] & & \\
T_1\mathrm{Emb}(M,N) & T_2\mathrm{Emb}(M,N) \arrow[l] & \cdots \arrow[l] & T_r\mathrm{Emb}(M,N) \arrow[l] \\
 & L_2\mathrm{Emb}(M,N) \arrow[u] & & L_r\mathrm{Emb}(M,N) \arrow[u] \arrow[ul, dashed] \arrow[ull, dashed]
\end{tikzcd}$$

<u>Thm</u> (Goodwillie + Klein + Weiss)

$$\mathrm{Emb}(M,N) \longrightarrow T_r\mathrm{Emb}(M,N)$$

is $\big((r+1)(c-2) + 3\cdot\dim(N)\big)$ - connected

where $c = \dim(N) - \text{handle-dim}(M)$

Reformulation (Boavida de Brito - Weiss)

$$\begin{tikzcd}
T_r \operatorname{Emb}(M,N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Map}_{\mathrm{Fin}}\big(\operatorname{con}(M;r), \operatorname{con}(N;r)\big) \arrow[d] \\
T_1 \operatorname{Emb}(M,N) \arrow[r] & \operatorname{Map}_{\mathrm{Fin}}\big(\operatorname{con}^{\mathrm{loc}}(M;r), \operatorname{con}^{\mathrm{loc}}(N;r)\big)
\end{tikzcd}$$

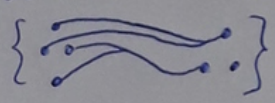
(derived)

Def:

$\partial M = \phi$

$\operatorname{con}(M;r) \to \mathrm{Fin}$ — configuration category of M — forget embedding into M

objects $= \coprod_{i=0}^{r} \operatorname{Emb}(\{1,\dots,i\}, M)$

morphisms $=$ 

$\operatorname{con}^{\mathrm{loc}}(M;r)$ — local configuration category

objects $=$ (morphisms in $\operatorname{con}(M;r)$ to a config. of size 1)

morphisms $=$ commuting triangles

$\partial M \neq \phi$

$\operatorname{con}(M;r) \to \mathrm{Fin}_*$

objects $= \coprod_{i=0}^{r} \operatorname{Emb}_*(\{0,1,\dots,i\}, M/\partial M)$ [ordered config. spaces relative to ∂M]

morphisms $=$ as above

With boundary:

extending $\operatorname{con}(\operatorname{collar}(\partial M);r) \longrightarrow \operatorname{con}(\operatorname{collar}(\partial N);r)$ induced by $\partial M \hookrightarrow \partial N$

$$\begin{tikzcd}
T_r \operatorname{Emb}_\partial(M,N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Map}^{\partial}_{\mathrm{Fin}_*}\big(\operatorname{con}(M;r), \operatorname{con}(N;r)\big) \arrow[d] \\
T_1 \operatorname{Emb}_\partial(M,N) \arrow[r] & \operatorname{Map}_{\mathrm{Fin}}\big(\operatorname{con}^{\mathrm{loc}}(\mathring{M};r), \operatorname{con}^{\mathrm{loc}}(\mathring{N};r)\big)
\end{tikzcd}$$

$(*)$

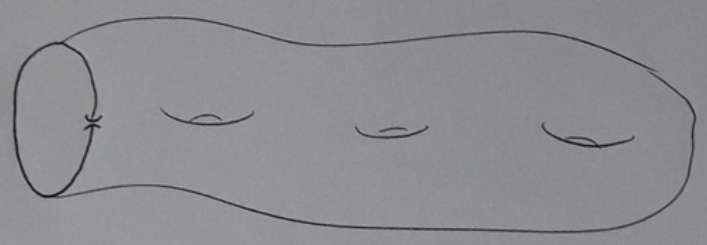
extending prescribed $\partial M \hookrightarrow \partial N$

Now set

$n \geq 3$

$$M = N = W^*_{g,1} := \#_g (S^n \times S^n) \setminus \left(\mathrm{int}(D^{2n}) \cup \{z\} \right)$$

$z \in \partial D^{2n}$



Then $c = 2n - n = n$.

<u>Rmk</u>: Since $M = N$, $(*)$ is a diagram of <u>E_1-spaces</u>.

<u>Lemma</u>: $\mathrm{Diff}_\partial(W^*_{g,1}) \hookrightarrow \mathrm{Emb}_\partial(W^*_{g,1}, W^*_{g,1})$ is a $\simeq$.

$\hookrightarrow \mathrm{Emb}_\partial(W^*_{g,1}, W^*_{g,1})$ is grouplike.

So for purposes of approximating this space, we may restrict to <u>invertible</u> components in $(*)$.

<u>Also</u>: The bottom row of $(*)$ may be simplified / rewritten

(h-principle / <u>linear</u> functors)

$$T_1 \mathrm{Emb}_\partial(M,N) \simeq \text{formal immersions} \simeq \mathrm{hAut}_*(W^*_{g,1}) \ltimes \mathrm{Map}_*(W^*_{g,1}, SO(2n))$$

$$\mathrm{Map}_{\mathrm{Fin}}(\mathrm{can}^{\mathrm{loc}}(\mathring{M};\nu), \mathrm{can}^{\mathrm{loc}}(\mathring{N};\nu)) \simeq \mathrm{hAut}_*(W^*_{g,1}) \ltimes \mathrm{Map}_*(W^*_{g,1}, \mathrm{hAut}_{\mathrm{Fin}}(\mathrm{can}(\mathbb{R}^{2n};\nu)))$$

Delaping, we get:

$$\begin{tikzcd}
B\mathrm{Diff}_\partial(W_{g,1}^*) \arrow[r] \arrow[d] & B\mathrm{hAut}^\partial_{\mathrm{Fin}_*}(\mathrm{con}(W_{g,1}^*; r)) \arrow[d] \\
B\left(\mathrm{hAut}_*(W_{g,1}^*) \ltimes \mathrm{Map}_*(W_{g,1}^*, SO(2n))\right) \arrow[r] & B\left(\mathrm{hAut}_*(W_{g,1}^*) \ltimes \mathrm{Map}_*(W_{g,1}^*, \mathrm{hAut}_{\mathrm{Fin}}(\mathrm{con}(\mathbb{R}^{2n}; r)))\right)
\end{tikzcd}$$

is <u>homotopy k-cartesian</u>, where

$$k = \underbrace{(n-2)}_{\text{positive!}}r - n + 2 .$$

"surreal"

② Exotic Pontrjagin classes.

$\mathbb{Q}$ coeff

$$\begin{tikzcd}
\mathbb{Q}[p_1, \dots, p_n] \cong H^*(BO(2n)) & H^*(B\mathrm{Homeo}(\mathbb{R}^{2n})) \arrow[l] \arrow[r, equal, "??"] & {} \\
\mathbb{Q}[p_1, p_2, \dots] \cong H^*(BO) \arrow[u] & H^*(BTOP) \arrow[l, "\cong"] \arrow[u] &
\end{tikzcd}$$

$|p_i| = 4i$

Sullivan
Kirby–Siebenmann

char. classes for $\mathbb{R}$ vector bundles

ch. classes for fibre bundles with Euclidean fibre

Thm (Weiss) For all $k \leq$ approx. $\frac{5}{4}n$

$$\boxed{p_{n+k} \neq 0 \quad \text{in} \quad H^*(B\mathrm{Homeo}(\mathbb{R}^{2n}))}$$

Relation to $\mathrm{Diff}_\partial(D^d)$:

$(d \neq 4)$

$$\{\text{smooth str. on } D^d\} \overset{\text{smoothing theory}}{\simeq} \mathrm{Map}_\partial(D^d, \underbrace{\{\text{smooth str. on } \mathbb{R}^d\}}_{\cong\ \mathrm{Homeo}(\mathbb{R}^d)/\mathrm{Diff}(\mathbb{R}^d)\ \simeq\ \mathrm{Homeo}(\mathbb{R}^d)/O(d)})$$

$$\cong \mathrm{Homeo}_\partial(D^d)/\mathrm{Diff}_\partial(D^d) \simeq B\mathrm{Diff}_\partial(D^d)$$

$$\rightsquigarrow \quad B\mathrm{Diff}_\partial(D^d) \simeq \Omega^d_0\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d)\right)$$

Morlet lemma

$$\Omega^{d+1}\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d)\right) \longrightarrow \Omega^{d+1}BO(d) \longrightarrow \Omega^{d+1}B\mathrm{Homeo}(\mathbb{R}^d) \longrightarrow B\mathrm{Diff}_\partial(D^d)$$

$$\Omega^{d+1}\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d)\right) \cong \mathrm{Diff}_\partial(D^d)$$

Thm (Weiss) The element p_{n+k} above evaluate to $\neq 0$ on $\pi_{4k+4n}(B\mathrm{Homeo}(\mathbb{R}^{2n})) \otimes \mathbb{Q}$

Cor $\boxed{\pi_{2n+4k-1}(B\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q} \neq 0 \quad \text{for } k \leq \sim \frac{5}{4}n}$

$\rightsquigarrow$ Recently gen. to <u>arbitrarily large k</u> by [Kupers - Randal-Williams]

K classes

smooth mfld bundle

$$\begin{tikzcd} \mathbb{R}^d \arrow[r] & T^v E \arrow[d] \\ F^d \arrow[r] & E \arrow[d] \\ & M \end{tikzcd}$$

$$x \in H^i(BO(d)) \rightsquigarrow \kappa(x) \in H^{i-d}(B\mathrm{Diff}(F))$$

$$x(T^v E) \in H^i(E) \rightsquigarrow \kappa(x)(E) := \int_F x \in H^{i-d}(M)$$

universally:

$$\begin{tikzcd} \mathbb{R}^d \arrow[r] & E^v \mathrm{Diff}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & EO(d) \arrow[d] \\ F \arrow[r] & E\mathrm{Diff}(F) \arrow[r] \arrow[d] & BO(d) \arrow[dl, bend left, "\kappa"] \\ & B\mathrm{Diff}(F) & \end{tikzcd}$$

& similarly for TOP bundles:

$$\begin{tikzcd} \mathbb{R}^d \arrow[r] & E^v \mathrm{Homeo}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E\mathrm{Homeo}(\mathbb{R}^d) \arrow[d] \\ F \arrow[r] & E\mathrm{Homeo}(F) \arrow[r] \arrow[d] & B\mathrm{Homeo}(\mathbb{R}^d) \arrow[dl, bend left, "\kappa"] \\ & B\mathrm{Homeo}(F) & \end{tikzcd}$$

Key defining property: if $x \in H^i(BO(d))$ or $H^i(B\mathrm{Homeo}(\mathbb{R}^d))$

$$\begin{tikzcd} \mathbb{R}^d \arrow[r] & T^v E \arrow[d] \\ F^d \arrow[r] & E^i \arrow[d] \\ & M \end{tikzcd}$$

then $\langle x(TE), [E] \rangle = \langle \kappa(x)(E), [M] \rangle$.

<u>Aim:</u> For $k \leq \sim \frac{s}{4} n$, find $z \in H_{2n+4k}(BHomeo(W_g))$

$W_g = \#_g (S^n \times S^n)$

such that
- $\langle \kappa(\mathcal{L}_{n+k}), z \rangle \neq 0$ — Hirzebruch L-poly.
- $\langle \kappa(P), z \rangle = 0$ — any <u>decomposable</u> monomial in P. classes.

<u>How it finishes the proof:</u>

Since $H_*(X;\mathbb{Q}) \cong \pi_*^s(X) \otimes \mathbb{Q} \cong \Omega_*^{fr}(X) \otimes \mathbb{Q}$,

represent z by $M \xrightarrow{f} BHomeo(W_g)$

smooth, closed, TM stably trivial.

$$\begin{tikzcd}
\mathbb{R}^{2n} \arrow[r] & T^\vee E \arrow[d] \\
W_g \arrow[r] & E \arrow[d] \\
 & M \arrow[r, "f"] & BHomeo(W_g)
\end{tikzcd}$$

$$\begin{aligned}
0 \neq \langle \kappa(\mathcal{L}_{n+k}), f([M]) \rangle &= \langle \kappa(\mathcal{L}_{n+k})(E), [M] \rangle \\
&= \langle \mathcal{L}_{n+k}(TE), [E] \rangle \\
&= \langle \mathcal{L}_{n+k}(T^\vee E), [E] \rangle \qquad TE \cong T^\vee E \oplus TM.
\end{aligned}$$

$$0 = \cdots\cdots = \langle P(T^\vee E), [E] \rangle$$

$\Rightarrow \quad P_{n+k}(\underbrace{T^\vee E}_{\text{rank}=2n}) \neq 0 \qquad \Rightarrow \quad P_{n+k} \neq 0$ in $H^*(BHomeo(\mathbb{R}^{2n}))$.

//.

To construct z : $\boxed{k \leq n}$

$$\begin{tikzcd}
H_* BDiff_\partial(W_{g,1}) \arrow[r] & H_* BDiff_\partial(W_{g,1}^*) \arrow[r] & H_* BHomeo(W_g) \\
x \arrow[r, mapsto] & y \arrow[u, bend left, "\varphi"] & z
\end{tikzcd}$$

$2n+4k$

[GRW]

emb. calculus

For $g \gg 0$

$$H^* BDiff_\partial(W_{g,1}) \cong \mathbb{Q}[\kappa(c) \mid c \text{ monomial in } p_{n/4} \cdots\cdots p_{n-1}, e]$$

So for $\frac{n}{4} \leq a, b \leq n$ we can find x :

$$\langle \kappa(p_a p_b), x \rangle \neq 0$$

$$\langle \kappa(P), x \rangle = 0 \quad \forall \text{ other monomials } P$$

ξ_{n+k} = $\mathbb{Q}$-lin. comb^n of monomials in P. classes

Thm (Berglund-Bergström '18) Every coefficient in ξ_{n+k} is $\neq 0$ (!)

$$\Rightarrow \quad \langle \kappa(\xi_{n+k}), x \rangle \neq 0.$$

$\boxed{7}$ j

$$
\begin{tikzcd}
H^*(BTOP) \arrow[d, phantom, "\cup\!\!\mid"] & & & \\
p \in (\text{decomposables}) \arrow[rr, dotted, "\varkappa"] \arrow[dd] & & BDiff_\partial(W^*_{g,1}) \arrow[r] \arrow[d] \arrow[dl, bend left] \arrow[uul, mapsto, "\varkappa(p)"'] \arrow[uul, bend left, "\varkappa(p)"] & BhAut^\partial_{Fin_*}(\mathrm{con}(W^*_{g,1};r)) \arrow[r] \arrow[d] & \tau_{\leq b} BhAut_\partial(W_{g,1}) \arrow[d] \arrow[dl, mapsto, "\varkappa(h_{n+1})"'] \\
 & & B(hAut_*(W^*_{g,1}) \ltimes Map_*(W^*_{g,1}, SO(2n))) \arrow[r] & B(hAut_*(W^*_{g,1}) \ltimes Map_*(W^*_{g,1}, hAut_{Fin}(\mathrm{con}(\mathbb{R}^{2n};r)))) \arrow[r] & \tau_{\leq b} BhAut_*(W_{g,1}) \\
(\text{decomposables}) \arrow[rr, dotted, "\varkappa"] \arrow[d, phantom, "\cap\!\!\mid"] & & B(hAut_\partial(W^*_{g,1}) \ltimes Map_*(W^*_{g,1}, SO(2n))) \arrow[ur] & & \\
H^*(BSO(2n)) \arrow[d, phantom, "\circlearrowleft"] & & & & \\
q^* & & & &
\end{tikzcd}
$$

$\circlearrowleft_\varphi$ $\quad$ $\underline{\underline{((n-2)r-n+2)\text{-cartesian}}}$ $\quad$ $\circlearrowleft_{id}$ $\quad$ $\circlearrowleft_f$ $\quad$ $\circlearrowleft_f$ $\quad$ $\circlearrowleft_f$

Postnikov truncation $b = (n-2)r - n - 1$

in practice this imposes restrictions on the possible range of degrees in which things work

<u>Pretend</u> (!) we have localised everywhere.

$BSO(2n)_{\mathbb{Q}} \simeq$ product of EM spaces

$\Rightarrow \exists!\ q : BSO(2n)_{\mathbb{Q}} \longrightarrow BSO(2n)_{\mathbb{Q}}$

with $q^*(p_b) = 0$

q^* acts trivially on the other p_i and on e

$\longrightarrow$ induces f (after rationalisation)

f (bottom row) and id (top-right) assemble to induce φ <u>in the range in which the square is cartesian</u>.

$\Rightarrow$ - $\langle \kappa(\xi_{n+k}), \varphi_*(y) \rangle = \langle \kappa(\xi_{n+k}), y \rangle \neq 0$

- P any <u>decomposable</u> monomial,

$$\begin{aligned} \langle \kappa(P), \varphi_*(y) \rangle &= \langle \kappa(P), f_*(y) \rangle \\ &= \langle f^* \kappa(P), y \rangle \\ &= \langle \kappa(g^* P), y \rangle \qquad (g^* P \neq P_a P_b) \\ &= 0 . \end{aligned}$$

$\Rightarrow$ Push $\varphi(y)$ forward to obtain $z \in H_* \mathrm{BHomeo}(W_g)$. //.