

Homological stability for configuration spaces on closed manifolds

Martin Palmer-Anghel // GeMAT seminar, IMAR // April–July 2025

Series abstract.

A classical result in algebraic topology — due to Arnol'd, McDuff and Segal in the 1970s — says that unordered configuration spaces on connected, non-compact manifolds are homologically stable. Here, the n th *unordered configuration space* $C_n(M)$ on a manifold M is the space of finite subsets $c \subset M$ of size n , topologised as a subquotient of the product M^n , and *homologically stable* means that $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ when n is sufficiently large as a function of i .

If M is instead a connected, *closed* manifold then its configuration spaces are generally *not* homologically stable: considering spherical braid groups one may check that it fails already in homological degree 1 for $M = S^2$. However, a number of more subtle stability or periodicity patterns have been discovered in the homology of configuration spaces on closed manifolds, depending in particular on the characteristic of the coefficient ring R that we use for homology. For example, in an appropriate range $n \gg i$, the homology $H_i(C_n(M); R)$ is:

- stable if $R = \mathbb{Q}$ or $R = \mathbb{F}_2$,
- stable if M is odd-dimensional and $R = \mathbb{Z}$,
- p -periodic if M is even-dimensional and R is a field of odd prime characteristic p .

A variety of different techniques have been used to prove these results, including *representation stability*, *scanning maps*, *replication maps*, *homology operations*, *factorisation homology* and *semi-simplicial resolutions*. The goal of this series of talks is to explain these different approaches.

Talks.

1. **10 April 2025** — We will follow the article
[O. Randal-Williams, *Homological stability for unordered configuration spaces*, Quart. J. Math. 64 \(2013\), pp. 303–326](#)
and prove homological stability with coefficients in \mathbb{Q} or \mathbb{F}_2 using semi-simplicial resolutions and transfer maps.
- 2,3. **28 April + 9 May 2025** — We will follow the article
[M. Bendersky, J. Miller, *Localization and homological stability of configuration spaces*, Quart. J. Math. 65 \(2014\), pp. 807–815](#)
and use scanning maps, localisations and obstruction theory to prove rational homological stability, as well as more complicated stability-like patterns in p -local homology.
- 4,5. **16 May + 23 May 2025** — We will follow the articles
[F. Cantero, M. Palmer, *On homological stability for configuration spaces on closed background manifolds*, Doc. Math. 20 \(2015\) pp. 753–805](#)
and
[A. Kupers, J. Miller, *Sharper periodicity and stabilization maps for configuration spaces of closed manifolds*, Proc. Amer. Math. Soc. 144 \(2016\) pp. 5457–5468](#)
and prove homological stability with coefficients in \mathbb{Z} when M is odd-dimensional and certain homological periodicity results when M is even-dimensional, using scanning maps, replication maps and homology operations for E_n -algebras.
6. **11 July 2025** — We will follow the article
[T. Church, *Homological stability for configuration spaces of manifolds*, Invent. Math. 188 \(2012\) pp. 465–504](#)
and prove rational homological stability using the concept of *representation stability* invented by T. Church and B. Farb.

7. **15 July 2025** — We will follow the article
B. Knudsen, *Betti numbers and stability for configuration spaces via factorization homology*,
Algebr. Geom. Topol. 17 (2017) pp. 3137–3187
and prove rational homological stability using factorisation homology and Lie algebra models.
We will also see some explicit calculations of rational homology that this method affords.
8. **16 July 2025** — We will follow the article
O. Randal-Williams, *Configuration spaces as commutative monoids*,
Bull. Lond. Math. Soc. 56 (2024) pp. 2847–2862
and prove rational homological stability by considering a commutative monoid structure on
one-point compactifications of configuration spaces.