Asymptotic monopoles, infinite-type surfaces and exotic groups

Martin Palmer-Anghel

22 December 2025

IMAR
Lucian Bădescu Prize Lecture

Overview

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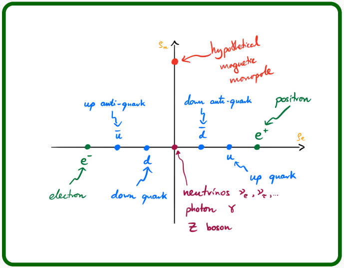
- ◇ Asymptotic magnetic monopoles



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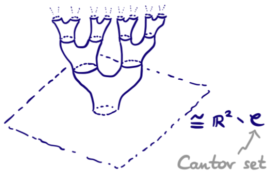

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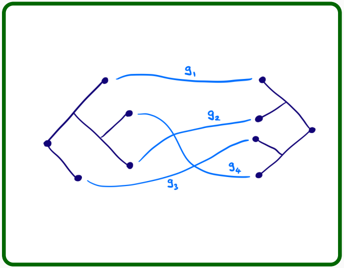

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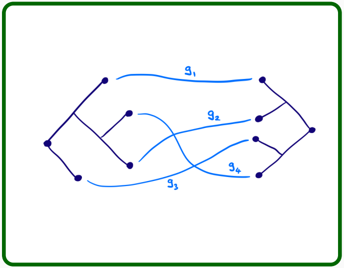

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Disclaimer: This is a *small* selection of mathematical domains in which homological stability is important.

Magnetic monopoles

Magnetic monopoles — Maxwell's equations

Maxwell's equations of electromagnetism (1865)

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Maxwell's equations of electromagnetism (1865) Formulation of Heaviside (1884)

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_e \qquad -\nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}_e$$

$\mathbf{E}/\mathbf{B}$: field ρ : charge density $\mathbf{j}$: current density

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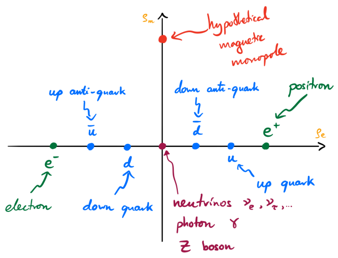

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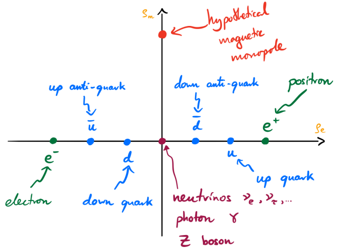
Dirac (1931):

- $\exists$ magnetic monopole $\Rightarrow$ quantisation of electric charges



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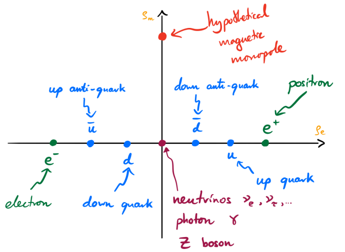

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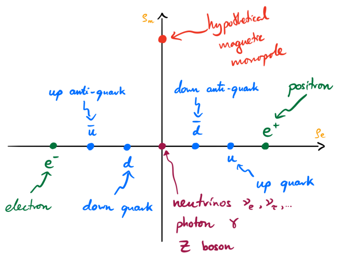

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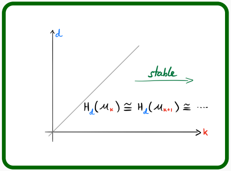

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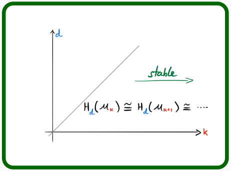

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Theorem (Cohen-Cohen-Mann-Milgram 1991):

$H_d(\mathcal{M}_k) \cong H_d(B_{2k})$ $\qquad$ (B_{2k} = braid group on $2k$ strands)

Magnetic monopoles — asymptotic monopoles

Kottke-Singer (2022):

- Partial compactification $\overline{\mathcal{M}}_k$ of $\mathcal{M}_k$

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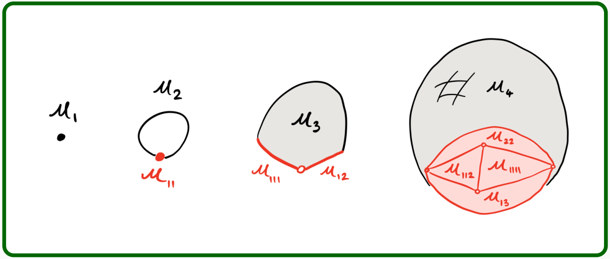

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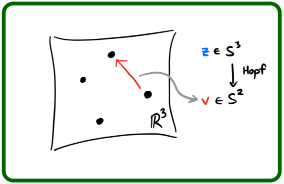

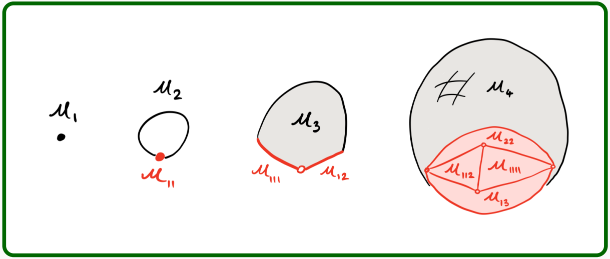

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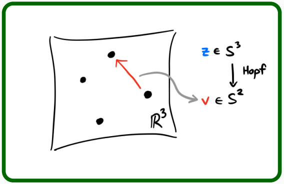

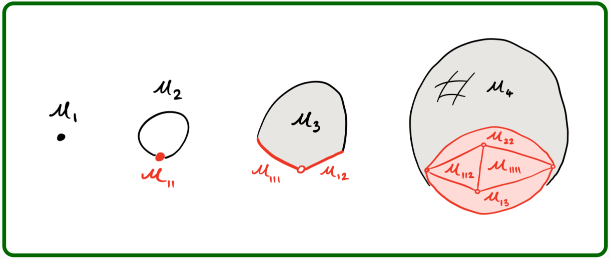

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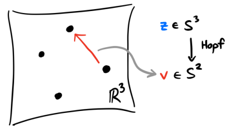

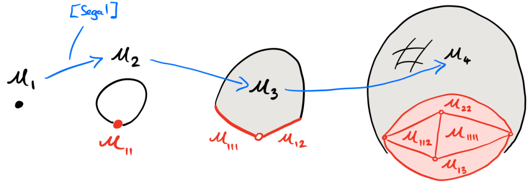

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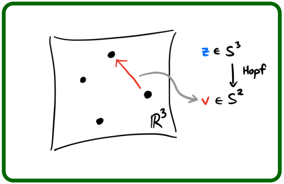

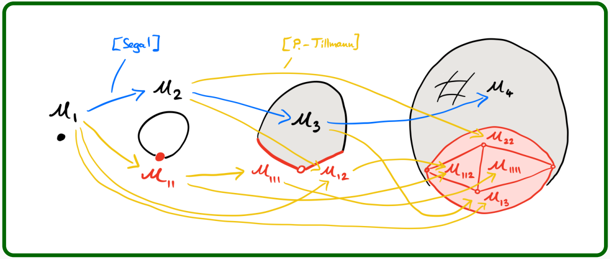

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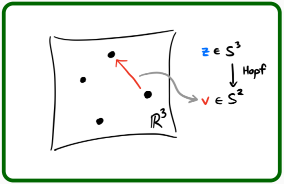

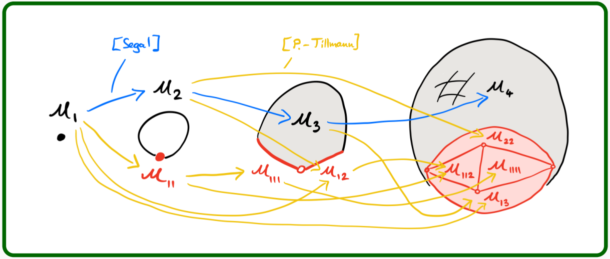

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Work in progress: Calculation of the *stable homology* $\lim_{k\to\infty} H_d(\mathcal{M}_{\lambda[k]})$

Infinite-type surfaces

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Theorem (Kerékjártó 1923 / Richards 1963):
Connected, orientable surfaces S are classified by:

- $g(S)$ *genus*
- $\mathcal{E}(S)$ *space of ends*
- $\mathcal{E}_{\mathrm{np}}(S)$ *subspace of non-planar ends*

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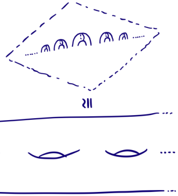

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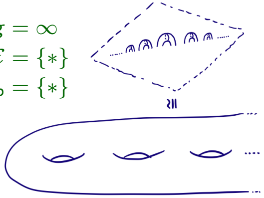
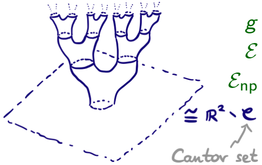


Punctured Cantor tree surface

$g = 0$
$\mathcal{E} = \mathcal{C}$
$\mathcal{E}_{\text{np}} = \varnothing$



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Theorem (Kerékjártó 1923 / Richards 1963):
Connected, orientable surfaces S are classified by:

- ◇ $g(S)$ *genus*
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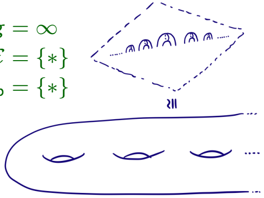

Infinite type surfaces arise from:

Punctured Cantor tree surface

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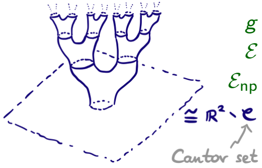

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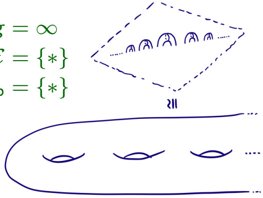

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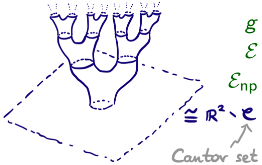

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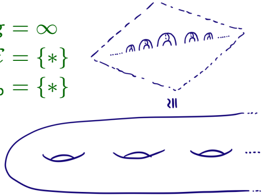

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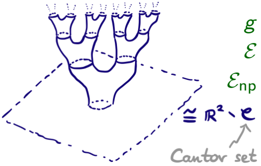

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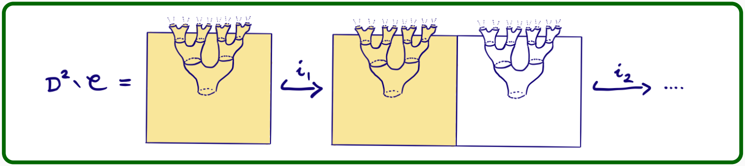

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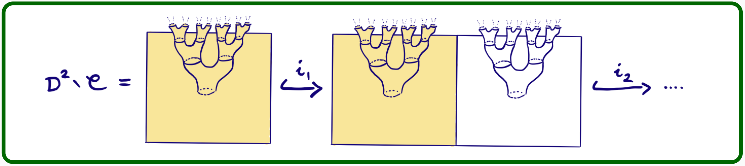

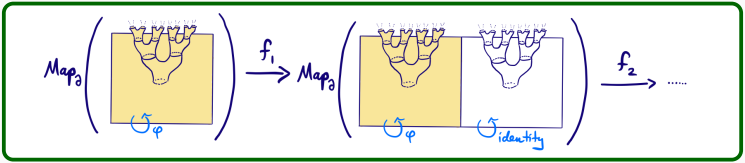

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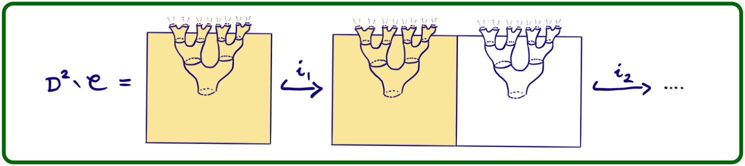

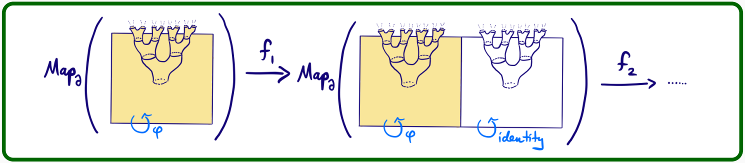

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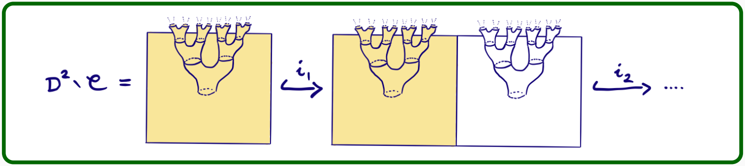

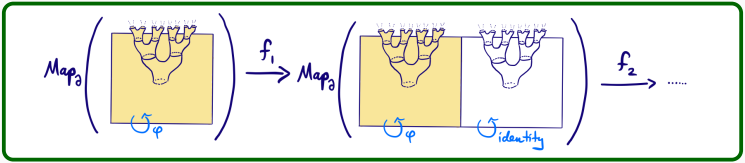

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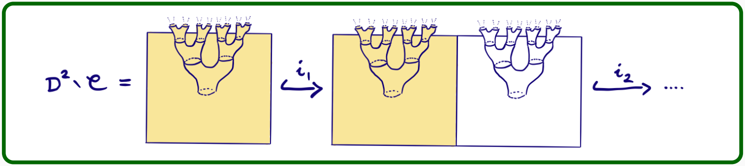

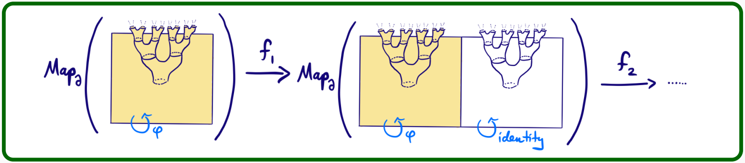

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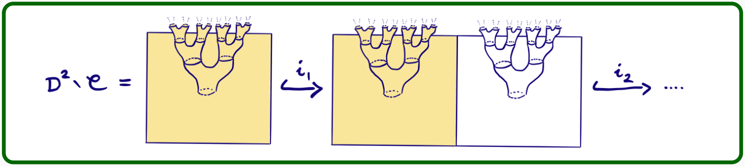

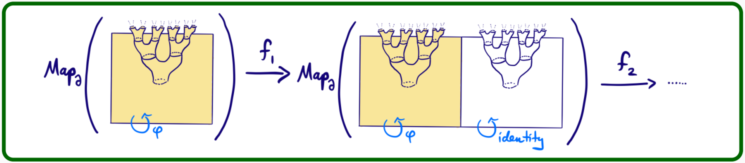

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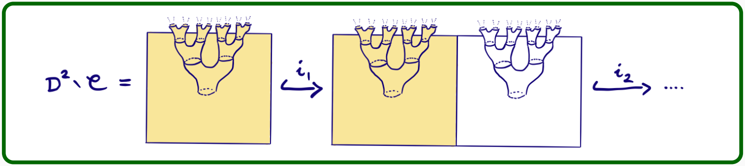

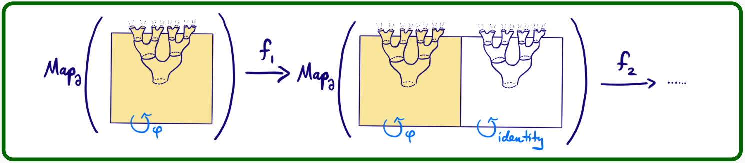

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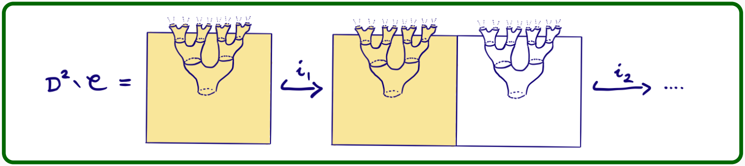

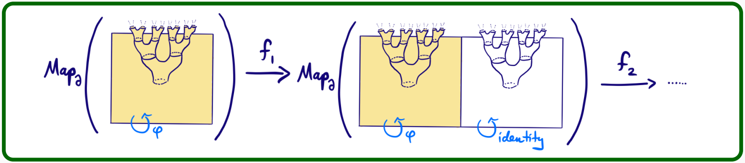

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- Short algebraic proof of Atiyah's L^2-index theorem
  (Chatterji-Mislin 2003)

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Ansatz: Every group of type $\mathcal{P}$ embeds into an acyclic group of type $\mathcal{P}$.

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Embeddings into acyclic groups — finiteness properties

A group G is of *type* F_n if there exists a $K(G,1)$ with finite n-skeleton

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Embeddings into acyclic groups — Thompson groups II

Construction of $\mathsf{V}(G)$:

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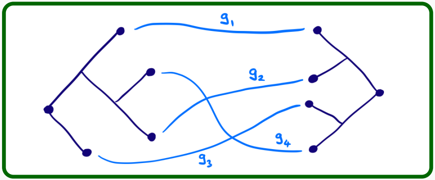
Embeddings into acyclic groups — Thompson groups II



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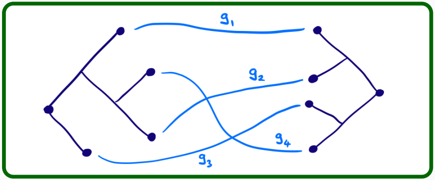

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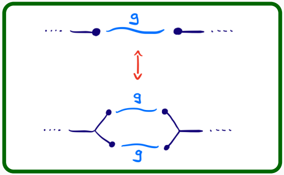

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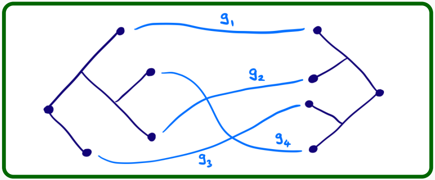

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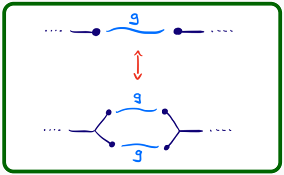

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Embeddings into acyclic groups — Thompson groups II

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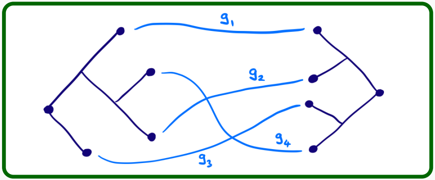

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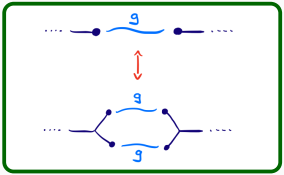

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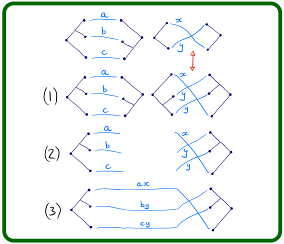

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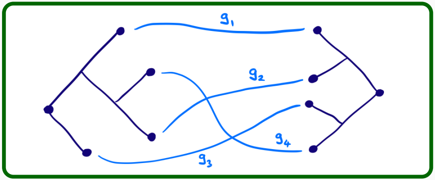

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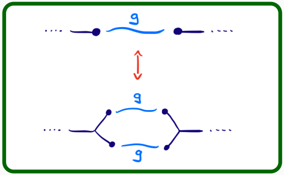

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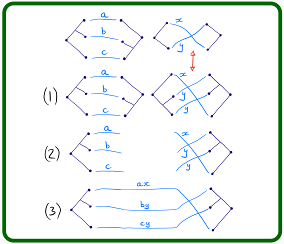

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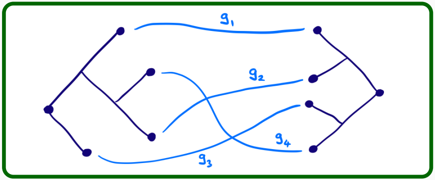

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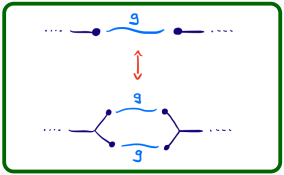

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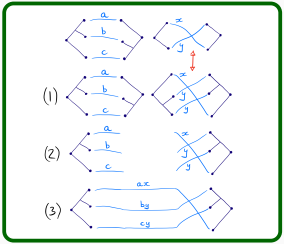

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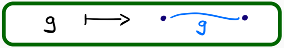

Embeddings into acyclic groups — Thompson groups III

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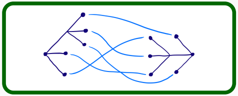

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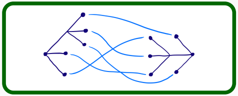

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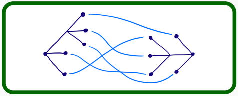

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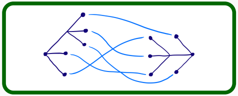

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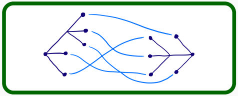

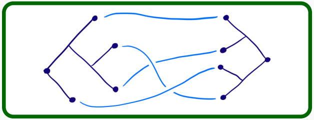

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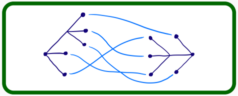

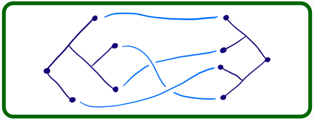

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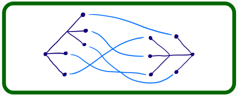

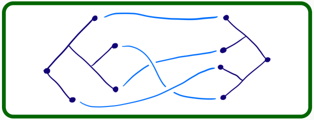

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