Embedding groups into acyclic groups via étale groupoids

Joint work with
Xiaolei Wu (Shanghai)

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Plan

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I Embeddings into acyclic groups

Def A group G is <u>acyclic</u> if $\tilde{H}_*(G) = 0$

Example (Baumslag - Gruenberg / Epstein '67/'68):

Commutator subgroup of $\langle a, b \mid a = [a, ba^{-1}b^{-1}][a, b^{-1}ab] \rangle$

[Baumslag - Gruenberg : combinatorial construction
Epstein : geometric construction (π_1 of grope construction)
Berrick - Wong (2004): they are the same]

Example (Mather '71) $\mathrm{Homeo}_c(\mathbb{R}^n)$

compact support

Theorem (Kan - Thurston '76)

Every countable group embeds into an <u>acyclic</u> countable group.

<u>Idea</u>: (1) $G \longrightarrow \underbrace{G^{\mathbb{Q}} \rtimes \mathrm{Aut}_c(\mathbb{Q})}_{\text{acyclic}}$

same method as [Mather]

(2) colimit argument $\longrightarrow$ <u>countable</u> subgroup containing G and still acyclic

Corollaries

- $\forall$ connected space X [Kan-Thurston '76]
  $\exists$ group G : $H_*(X) \cong H_*(G)$

  $\longrightarrow$ Idea: • CW-approximation X' of X
  • mirror the cell structure of X' using groups

  $\left(\text{embedding into an acyclic group}\right) \longleftrightarrow \left(\text{cone of a space}\right)$

- short proof of Atiyah's L^2-index theorem for countable coverings of Riemannian manifolds [Chatterji-Mislin '03]

  $\longrightarrow$ Idea: • The statement is natural w.r.t. inclusions of (countable) deck transformation groups.
  • It is easy to check if the deck transformation group is acyclic.
  • Apply [Kan-Thurston].

Def A group G has type F_n if it has a classifying CW-complex (i.e. $K(G,1)$ complex) with finite n-skeleton.

$F_1 \longleftrightarrow$ finitely generated
$F_2 \longleftrightarrow$ finitely presented

Observation $F_n \Rightarrow F_{n-1}$

Thm (Bieri '76) (Stallings '63 $n=3$)

Stallings–Bieri group

$F_n \not\Leftarrow F_{n-1}$

Specifically, $SB_n = \ker\big((F_2)^n \to \mathbb{Z}\big)$ has type F_{n-1} but <u>not</u> type F_n.

each gen. $\mapsto 1$

Thm (Skipper–Witzel–Zaremsky '17)

$\exists$ <u>simple</u> group of type F_{n-1} but not F_n

Thm A (P.–Wu '25)

$\exists$ <u>simple</u>, <u>acyclic</u> group of type F_{n-1} but not F_n

<u>Ansatz</u>: Every group of type P embeds into an acyclic group of type P.

True for:

$P = \phi$
$P =$ countable } [Kan–Thurston '76]

$P =$ type F_1 [Baumslag–Dyer–Heller '80]
$P =$ type F_2 [Baumslag–Dyer–Miller '83]

Thm B (P.–Wu '25)

True for $P =$ type F_n, for each $n \geq 1$
(and $n = \infty$)

(II) Thompson groups & variants

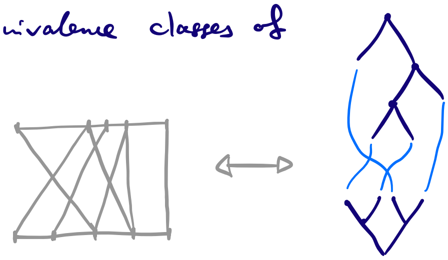
Fix $d \geqslant 2$

d^{th} Higman–Thompson group

All trees planar & rooted.

$V_d := \{ \varphi \in \mathrm{Homeo}(\mathfrak{C}) \mid \varphi$ given by
- cut $\mathfrak{C}$ into fin. many d-adic pieces
- rearrange + rescale by factors of d $\}$

= equivalence classes of



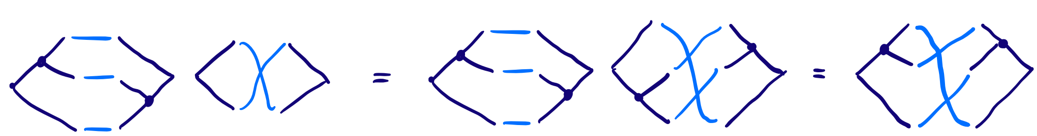
finite d-ary trees

bijection $\in \Sigma_n$

eq. relation generated by

Composition in $\mathrm{Homeo}(\mathfrak{C})$

$\longleftrightarrow$
- expand until middle trees are the same
- compose bijections



Timeline

$\langle a,b,c,d \mid \underbrace{b^a=b^2,\ c^b=c^2,\ d^c=d^2,\ a^d=a^2}_{\text{max. normal}} \rangle$

1951 (Higman) $\exists$ infinite, fin. pres., simple group ("Higman group")

1965 (Thompson) $V = V_2$ ——— " ———
type F_∞

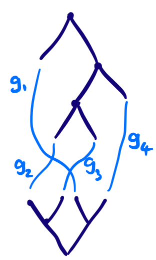
1974 (Higman) V_d type F_∞
d even : simple
d odd : $V_d \underset{(2)}{\trianglerighteq} [V_d, V_d]$ simple

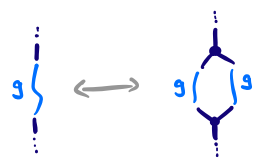
[Brown '87]

Variants : Thompson-like groups

① Labelled Higman-Thompson groups

Replace Σ_n with $G \wr \Sigma_n = G^n \rtimes \Sigma_n$ $\rightsquigarrow$ $V_d(G)$





Note G embeds into $V_d(G)$ via $g \longmapsto$ $|g$

Theorem [Wu-Wu-Zhao-Zhou, 2024]

G is of type F_n
$\Updownarrow$
$V_2(G)$ is of type F_n

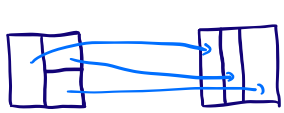
② Brin-Thompson groups [Brin, 2004]

Replace $\mathfrak{C}$ with $\mathfrak{C}^k$, $k \geq 1$

$\varphi \in kV_d$ acts on $\mathfrak{C}^k$ by

- cutting it into finitely many bricks
- rearrange + rescale by factors of d

cuboids of side-lengths $1/d^n$ & vertices = d-adic rationals

Eg  $\in 2V = 2V_2$

- kV is fin. presentable & simple [Brin, Hennig-Matucci]
- $kV \not\cong \ell V$ for $k \neq \ell$ [Brin, Bleak-Lanoue]

③ Twisted Brin–Thompson groups [Belk–Zaremsky, 2020]

Replace $\mathfrak{C}$ with $\mathfrak{C}^S$ (S any set)

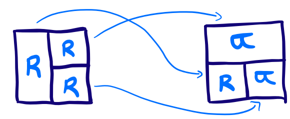
Choose $G \subseteq \mathrm{Sym}(S)$

- Bricks have length 1 in all but fin. many dimensions
- Rearrange, rescale and $\underbrace{\text{rotate/reflect}}$ each brick.

permute its coordinates by any $g \in G$.

$\leadsto$ SV_d^G

Eg



Theorem [Belk-Zaremsky, 2020]:

- SV_2^G is simple
- For certain choices of $G \subset \mathrm{Sym}(S)$, SV_2^G is of type F_{n-1} but not of type F_n.

Thm C (P.–Wu '25)

(i) For any group G, $V_2(G)$ is acyclic.

(ii) For any subgroup $G \subseteq \mathrm{Sym}(S)$, SV_2^G is acyclic.

Corollaries

(i) + [Wu-Wu-Zhao-Zhou] $\Rightarrow$ Thm B

and $V_2(SB_n)$ is acyclic & of type F_{n-1} but not type F_n

(ii) + [Belk-Zaremsky] $\Rightarrow$ Thm A

III Homology of Thompson-like groups

Thm (Brown '92) $\tilde{H}_*(V_2; \mathbb{Q}) = 0$

Thm (Szymik–Wahl '19) $H_*(V_d; \mathbb{Z}) \cong H_*(\Omega^\infty_\circ \underbrace{M(\mathbb{Z}/_{d-1})}_{\text{Moore spectrum}}; \mathbb{Z})$

Cor

$\tilde{H}_*(V_2; \mathbb{Z}) = 0$

$\tilde{H}_*(V_d; \mathbb{Q}) = 0 \quad \forall d \geq 2$

Thm (Li '22) $\forall k \geq 1$, $\left.\begin{array}{l}\tilde{H}_*(kV_2; \mathbb{Z}) = 0 \\ \tilde{H}_*(kV_d; \mathbb{Q}) = 0\end{array}\right\}$ Re-proven by [KLMMS'24]

Kupers
Lemann
Malkiewich
Miller
Sroka

Methods:

algebraic K-theory & homological stability

[SW] of a category of "Cantor algebras"

[Li] of étale groupoids

[KLMMS] of a category of "scissors congruences"

We use this approach for Thm C.

Work in progress (P.-Wu '26+)

$H_*(V_d(G); \mathbb{Z}) \cong H_*(\Omega^\infty_\circ \operatorname{hocofib}(\Sigma^\infty BG_+ \xrightarrow{d-1} \Sigma^\infty BG_+); \mathbb{Z})$

↳ acyclicity of $V_2(G)$

$\mathbb{Q}$-acyclicity of $V_d(G)$

$H_*(V_d(G); \mathbb{Z})$ depends only on $H_*(G; \mathbb{Z})$ as a coalgebra

$H_1(V_d(G); \mathbb{Z}) \cong \begin{cases} G^{ab}/(d-1) & d \text{ even} \\ G^{ab}/(d-1) \oplus \mathbb{Z}/2 & d \text{ odd} \end{cases}$ Can also be computed "by hand".

Def

Étale groupoid :

$$\overset{\text{inv.}}{\circlearrowleft}$$

$$G^{(0)} \xrightarrow{id} G \underset{t}{\overset{s}{\rightrightarrows}} G^{(0)} \qquad \begin{array}{c} G \times_{G^{(0)}} G \\ \downarrow \\ G \end{array}$$

continuous

s, t are local homeomorphisms

Examples

- discrete groups
- $\mathcal{V}_2$ "one-sided full shift"

$$\mathcal{V}_2^{(0)} = \{0,1\}^{\mathbb{N}} = X \qquad X \xrightarrow{T} X, \quad (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$$

$$\mathcal{V}_2 \subseteq X \times \mathbb{Z} \times X$$

$$\left\{ (x, n, y) \; : \; \exists\, \ell, m \geq 0 \; : \; \begin{array}{l} n = \ell - m \\ T^{\ell}(x) = T^{m}(y) \end{array} \right\}$$

Topological full group

$F(G)$ = group of homeo of of $G^{(0)}$ that locally look like elements of G

Ex $F(\mathcal{V}_2) \cong V_2$

Groupoid homology [Crainic – Moerdijk '99]

Ex $H_*(\mathcal{V}_2) = 0$ — exercise using excision

More generally

$(d-1)$-torsion

$$H_*(\mathcal{V}_d ; A) \cong \begin{cases} A/(d-1) & * = 0 \\ A_{(d-1)} & * = 1 \\ 0 & * \geq 2 \end{cases}$$

[Matui '12]

Construction (Li '22)

G étale groupoid

$G^{(0)}$ locally compact Hausdorff
totally disconnected

$\longrightarrow$ symmetric monoidal category $\mathcal{B}_G$

$\longrightarrow$ spectrum $K\mathcal{B}_G$

Theorem (Li '22)

if G is purely infinite minimal — $\forall$ cpt open $U, V \subseteq G^{(0)}$, $V \neq \emptyset$

$G^{(0)}$ has no isolated points — $\exists$ cpt open bisection $\sigma \subseteq G$ with $s(\sigma) = U$, $r(\sigma) \subseteq V$.

then $H_*(G) \cong \tilde{H}_*(K\mathcal{B}_G)$

[Crainic - Moerdijk]

$$H_*(F(G)) \cong H_*(\Omega_0^\infty K\mathcal{B}_G)$$

group homology

hom. stab^y hidden in proof

Coro (via Hurewicz Thm)

$H_*(G) = 0 \implies F(G)$ is acyclic.

Coro (via Milnor-Moore Thm)

$H_*(G; \mathbb{Q})$ determines $H_*(F(G); \mathbb{Q})$

Example

$F(\mathcal{V}_2) \cong V_2$ is acyclic.

Thm C(ii)

For any subgroup $G \subseteq \mathrm{Sym}(S)$, SV_2^G is acyclic.

Sketch of proof

Let
$$S\mathcal{D}_2 = \begin{cases} \text{obj} = \prod_S \mathcal{D}_2^{(0)} \\ \text{mor} = \left\{ (x_s) \in \prod_S \mathcal{D}_2 \;\middle|\; \begin{array}{l} x_s \in \mathcal{D}_2^{(0)} \text{ for all but} \\ \text{fin. many } s \in S \end{array} \right\} \end{cases}$$

⚠ Do not give this the subspace topology induced from $\prod_S \mathcal{D}_2$. Instead: "relative product topology".

strictly finer whenever S is infinite

Basis given by $\prod_{s\in S} U_s$ for $U_s \subseteq \mathcal{D}_2$ open with $U_s = \mathcal{D}_2^{(0)}$ for all but fin. many s

The action $G \curvearrowright S$ induces $G \curvearrowright S\mathcal{D}_2$

$\longrightarrow S\mathcal{D}_2 \rtimes G$

Thm (P.-Wu '25) $F(S\mathcal{D}_2 \rtimes G) \cong SV_2^G$

This would be false if we used the subspace topology!

Thm [Matui '12] $\exists$ spectral sequence

$$H_*(G; H_*(S\mathcal{D}_2)) \Rightarrow H_*(S\mathcal{D}_2 \rtimes G)$$

Lemma Let $T = S \setminus \{s_0\}$. Then $S\mathcal{D}_2 \cong \mathcal{D}_2 \times T\mathcal{D}_2$.

- $H_*(\mathcal{D}_2) = 0$
- Künneth $\longrightarrow H_*(S\mathcal{D}_2) = 0$
- Matui $\longrightarrow H_*(S\mathcal{D}_2 \rtimes G) = 0$
- Li $\longrightarrow F(S\mathcal{D}_2 \rtimes G)$ is acyclic.

$$F(S\mathcal{D}_2 \rtimes G) \cong SV_2^G$$

$\square$

Extras

Proof of Thm C(i):

- $F(\mathcal{V}_2 \times G) \cong V_2(G)$
- Künneth $\rightsquigarrow H_*(\mathcal{V}_2 \times G) = 0$
- Li $\rightsquigarrow F(\mathcal{V}_2 \times G)$ acyclic.

Speculation about $\mathrm{br}V_d$

Replace Σ_n with B_n in the definition of V_d

Q: What is $H_*(\mathrm{br}V_d)$?

maybe define braided topological full groups ??

Note:

- $\mathrm{br}V_d$ is not a topological full group
- methods of Szymik–Wahl also do not apply — they assume a symmetric monoidal structure

Guess: $H_*(\Omega_0 \,\mathrm{hofib}(S^2 \xrightarrow{d-1} S^2))$

$\Downarrow$

$H_1(\mathrm{br}V_d) \cong \pi_2 \,\mathrm{hofib}(S^2 \xrightarrow{d-1} S^2)$

$\cong \mathbb{Z}/(d-1)$

Note:

- $\mathrm{br}V_d \twoheadrightarrow \mathbb{Z}/(d-1)$ writhe mod $d-1$
- $H_1(V_d) \cong \mathbb{Z}/(d-1) \otimes \mathbb{Z}/2$