Heisenberg homology on surface configurations

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Abstract

We study the action of the mapping class group of $\Sigma = \Sigma_{g,1}$ on the homology of configuration spaces with coefficients twisted by the discrete Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma)$, or more generally by any representation V of $\mathcal{H}$. In general, this is a twisted representation of the mapping class group $\mathfrak{M}(\Sigma)$ and restricts to an untwisted representation on the Chillingworth subgroup $\mathrm{Chill}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$. Moreover, it may be untwisted on the Torelli group $\mathfrak{T}(\Sigma)$ by passing to a $\mathbb{Z}$-central extension, and, in the special case where we take coefficients in the Schrödinger representation of $\mathcal{H}$, it may be untwisted on the full mapping class group $\mathfrak{M}(\Sigma)$ by passing to a double covering. We illustrate our construction with several calculations for 2-point configurations, in particular for genus-1 separating twists.

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Introduction

The braid group B_m was defined by Artin in terms of geometric braids in $\mathbb{R}^3$; equivalently, it is the fundamental group of the configuration space $\mathcal{C}_m(\mathbb{R}^2)$ of m unordered points in the plane. Another equivalent description is as the mapping class group $\mathfrak{M}(\mathbb{D}_m)$ of the closed 2-disc with m interior points removed. (The *mapping class group* of a surface is the group of isotopy classes of self-diffeomorphisms fixing the boundary pointwise.)

One of the earliest representations of B_m (not factoring through the projection onto the symmetric group) is the *Burau representation* [14] of B_m in $GL_m(\mathbb{Z}[t^{\pm 1}])$, which is most naturally defined as the induced action of $B_m = \mathfrak{M}(\mathbb{D}_m)$ on the twisted homology of $\mathbb{D}_m$. There is also a natural action of $\mathfrak{M}(\mathbb{D}_m)$ on configuration spaces $\mathcal{C}_n(\mathbb{D}_m)$; considering the induced action on the homology of these configuration spaces, Lawrence [26] extended

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this to define a representation of B_m for each $n \geq 1$. The $n = 2$ version is known as the *Lawrence-Krammer-Bigelow representation*, and a celebrated result of Bigelow [11] and Krammer [25] states that this representation of B_m is *faithful*, i.e. injective.

On the other hand, for almost all other surfaces Σ, the question of whether $\mathfrak{M}(\Sigma)$ admits a faithful, finite-dimensional representation over a field (whether it is *linear*) is open. The mapping class group of the torus is $SL_2(\mathbb{Z})$, which is evidently linear, and the mapping class group of the closed orientable surface of genus 2 was shown to be linear by Bigelow and Budney [12], as a corollary of the linearity of B_n. However, nothing is known in genus $g \geq 3$.

An and Ko studied in [1] extensions of the Lawrence-Krammer-Bigelow representation to homological representations of surface braid groups; see also [6]. Here, we are interested in the mapping class group interpretation of classical braids. In particular, inspired by the linearity question, one could ask about natural analogues of the Lawrence representations (including the Lawrence-Krammer-Bigelow representation) for the mapping class group $\mathfrak{M}(\Sigma)$ of the compact orientable surface $\Sigma = \Sigma_{g,1}$ of genus g with one boundary component. There is a natural action of $\mathfrak{M}(\Sigma)$ on the configuration spaces $\mathcal{C}_n(\Sigma)$; the key question is the choice of *local system* on $\mathcal{C}_n(\Sigma)$ for the coefficients of homology, which depends on choosing a quotient of its fundamental group.

In general, for any surface Σ and $n \geq 2$, the abelianisation of $\pi_1(\mathcal{C}_n(\Sigma))$ is canonically isomorphic to $H_1(\Sigma) \times C$, where C is a cyclic group of order ∞ if Σ is planar, of order $2n-2$ if $\Sigma = \mathbb{S}^2$ and of order 2 in all other cases (see for example [16, §6]). In the case $\Sigma = \mathbb{D}_m$, the abelianisation is $\mathbb{Z}^m \times \mathbb{Z}$, and the Lawrence representations are defined using the local system given by the quotient $\pi_1(\mathcal{C}_n(\mathbb{D}_m)) \twoheadrightarrow \mathbb{Z}^m \times \mathbb{Z} \twoheadrightarrow \mathbb{Z} \times \mathbb{Z}$, where the second quotient is given by addition in the first m factors. However, in the non-planar case (in particular if $\Sigma = \Sigma_{g,1}$), we *lose information* by passing to the abelianisation, since the cyclic factor C – which counts the self-winding or "writhe" of a loop of configurations – has order 2 rather than order ∞.

To obtain a better analogue of the Lawrence representations in the setting $\Sigma = \Sigma_{g,1}$ for $g > 0$, we consider instead a larger, non-abelian quotient of $\pi_1(\mathcal{C}_n(\Sigma))$, the discrete Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma)$. This is a 2-nilpotent group that arises very naturally as a quotient of the surface braid group $\pi_1(\mathcal{C}_n(\Sigma))$ by forcing a single element to be central. When $n \geq 3$ it is known by [4] to be the 2-nilpotentisation of the surface braid group, but for $n = 2$ it differs from the 2-nilpotentisation. The key property of this quotient is that it still detects the self-winding (or "writhe") of a loop of configurations *without reducing modulo* 2, at the expense of being non-abelian.

Main Theorem (Theorems 39, 41, 43 and 56). *Let $\Sigma = \Sigma_{g,1}$ for $g \geq 1$ and fix $n \geq 2$. Associated to any representation V of the discrete Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma)$, the action of $\mathfrak{M}(\Sigma)$ on the configuration space $\mathcal{C}_n(\Sigma)$ equipped with a local system with fibre V gives rise to:*

(A) *a twisted representation of the full mapping class group $\mathfrak{M}(\Sigma)$;*
(B) *a representation of the Chillingworth subgroup* $\mathrm{Chill}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$*;*
(C) *a representation of a $\mathbb{Z}$-central extension of the Torelli group $\mathfrak{T}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$.*

In the special case where V is the Schrödinger representation of $\mathcal{H}$, we moreover obtain:

(D) *a unitary representation of a $\mathbb{Z}/2$-central extension of the full mapping class group $\mathfrak{M}(\Sigma)$, called the metaplectic mapping class group $\widetilde{\mathfrak{M}}(\Sigma)$.*

Remark 1. A *twisted representation* of a group G is a functor $\mathrm{Ac}(G \curvearrowright X) \to \mathrm{Mod}_R$, where $\mathrm{Ac}(G \curvearrowright X)$ is the *action groupoid* of some G-set X. See §5.1 for more details. An ordinary (untwisted) representation corresponds to $X = \{*\}$, in which case $\mathrm{Ac}(G \curvearrowright X)$ is the group G.

We emphasise that only point (A) of the Main Theorem concerns twisted representations; points (B), (C) and (D) concern *untwisted* representations.

Remark 2. We emphasise also that these representations are computable. To demonstrate this, we calculate in §8 explicit matrices for these representations in the case when $n = 2$ and V is the regular representation of $\mathcal{H}$.

Remark 3. Each of the (twisted) $\mathfrak{M}(\Sigma)$-representations of the Main Theorem has kernel contained in the intersection of the nth term of the Johnson filtration and the kernel of the Magnus representation; see Proposition 59.

Remark 4. The metaplectic mapping class group is the unique non-trivial $\mathbb{Z}/2$-central extension of $\mathfrak{M}(\Sigma)$ when the genus is at least 4; see Remark 57.

Outline. In §1 we define and study the quotient $\mathcal{H}$ of the surface braid group. In §2 we study the Borel-Moore homology (with local coefficients) of configuration spaces on Σ, showing that it is a free module with an explicit free generating set. Next, in §3, we show that the action of the mapping class group on the surface braid group descends to $\mathcal{H}$. Then we study this induced action on $\mathcal{H}$ in detail, and in particular determine its kernel, as well as the subgroup of the mapping class group that acts by *inner* automorphisms under this action. The latter group turns out to be the Torelli group, whereas the kernel turns out to be the Chillingworth subgroup: the mapping classes that act trivially on homotopy classes of non-vanishing vector fields. In §4 we then describe a general trick (under certain conditions) for untwisting twisted representations of groups by passing to a central extension.

Section 5 puts all of this together and constructs the representations (A), (B) and (C) of the Main Theorem. In particular, §5.2 explains the notion of a *twisted representation* of a group, and constructs twisted representations of the full mapping class group. In §6 we explain point (D) of the Main Theorem, the construction of an untwisted representation of a double cover of the mapping class group $\mathfrak{M}(\Sigma)$ when we take coefficients in the Schrödinger representation. The untwisting in this case uses the *Segal-Shale-Weil representation* of the metaplectic group. We also discuss a variation of this construction using finite-dimensional analogues of the Schrödinger representation.

In §7 we discuss relations with the Moriyama and Magnus representations of mapping class groups, and deduce that the kernel of our representation is contained in $\mathfrak{J}(n) \cap \ker(\text{Magnus})$, where $\mathfrak{J}(*)$ is the Johnson filtration. In §8 we explain how to compute matrices for our representations with respect to the free basis coming from §2. We carry out this computation explicitly in the case of configurations of $n = 2$ points and where

$V = \mathbb{Z}[\mathcal{H}]$ is the regular representation of $\mathcal{H}$; this special case of our construction is the most direct analogue of the Lawrence-Krammer-Bigelow representation of B_m.

Representations of the Heisenberg group. Fix $n \geq 2$, a ring R and a $(\mathbb{Z}[\mathcal{H}], R)$-bimodule V (i.e. a left representation of the Heisenberg group $\mathcal{H}$ over R). Our Main Theorem constructs (over R) a twisted representation (A) of the mapping class group and untwisted representations (B) of the Chillingworth subgroup and (C) a $\mathbb{Z}$-central extension of the Torelli group.

The basic choice of input representation V is the regular representation of $\mathcal{H}$, in other words setting $R = \mathbb{Z}[\mathcal{H}]$ and $V = \mathbb{Z}[\mathcal{H}]$ as a $(\mathbb{Z}[\mathcal{H}], \mathbb{Z}[\mathcal{H}])$-bimodule. However, there are many other representations that one could choose, and in some cases the properties of specific choices of V allow us to refine our construction: for example, taking V to be the Schrödinger representation allows us to construct an untwisted representation (D) of a double cover of the full mapping class group.

We list here some representations V of $\mathcal{H}$ for which we expect the associated homological representations of the Main Theorem to be especially interesting. Some of these, notably the Schrödinger representation, are discussed in more detail in §6.

(1) The real Heisenberg group is frequently defined as a subgroup $\mathcal{H}_{\mathbb{R}} \subset GL_{g+2}(\mathbb{R})$. The discrete Heisenberg group $\mathcal{H} \subset \mathcal{H}_{\mathbb{R}}$ is then realised as a group of $(g+2) \times (g+2)$ matrices whose coefficients are integers, except the upper-right one which is a half integer. This gives a faithful representation of the discrete Heisenberg group $\mathcal{H}$ on the free abelian group $\frac{1}{2}\mathbb{Z} \oplus \mathbb{Z}^{g+1} \cong \mathbb{Z}^{g+2}$, which we call the *tautological representation*.

(2) The *Schrödinger representation*

$$\mathcal{H} \longrightarrow U\big(L^2(\mathbb{R}^g)\big) \tag{1}$$

is a unitary representation of $\mathcal{H}$ on a Hilbert space. The unitarity of this representation is preserved by our construction: the resulting homological representation of the metaplectic mapping class group is also unitary.

(3) The *finite dimensional Schrödinger representation*

$$\mathcal{H} \longrightarrow U\big(L^2((\mathbb{Z}/N)^g)\big) \qquad \text{(for } N \geq 2) \tag{2}$$

is a unitary representation of $\mathcal{H}$ on a finite-dimensional Hilbert space, which has a famous geometric interpretation as a space of theta functions. It can also be realised in the context of knot theory and TQFTs [19, 20, 18].

(4) For a positive integer N, let $\mathcal{H}_N = \mathcal{H}/\langle u^N \rangle$ be the quotient of the Heisenberg group $\mathcal{H}$ by the N-th power of a generator u of its centre. This is the *N-torsion Heisenberg group*. We may then compose the quotient map $\mathcal{H} \twoheadrightarrow \mathcal{H}_N$ with representations of $\mathcal{H}_N$, which are related to pairs of q-commuting matrices for $q = \exp\big(\frac{2\pi i}{N}\big)$.

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Contents

1 A non-commutative local system on configuration spaces of surfaces

Let $\Sigma = \Sigma_{g,1}$ be a compact, connected, orientable surface with genus $g \geq 1$ and one boundary component. For $n \geq 2$, we define the n-point unordered configuration space of Σ as

$$\mathcal{C}_n(\Sigma) = \{\{c_1, c_2, ..., c_n\} \subset \Sigma, c_i \neq c_j \text{ for } i \neq j\},$$

topologised as a quotient of a subspace of Σ^n. The surface braid group $\mathbb{B}_n(\Sigma)$ is then defined as $\mathbb{B}_n(\Sigma) = \pi_1(\mathcal{C}_n(\Sigma))$. We will use Bellingeri's presentation [3] for the surface braid group $\mathbb{B}_n(\Sigma)$:

- The generators are: $\sigma_1, \sigma_2, ..., \sigma_{n-1}, \alpha_1, \alpha_2, ..., \alpha_g, \beta_1, \beta_2, ..., \beta_g$.
- The relations among them are:
  (BR1) $[\sigma_i, \sigma_j] = 1$ for $|i - j| \geq 2$;
  (BR2) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i - j| = 1$;
  (CR1) $[\alpha_r, \sigma_i] = [\beta_r, \sigma_i] = 1$ for $i > 1$;
  (CR2) $[\alpha_r, \sigma_1\alpha_r\sigma_1] = [\beta_r, \sigma_1\beta_r\sigma_1] = 1$;
  (CR3) $[\alpha_r, \sigma_1^{-1}\alpha_s\sigma_1] = [\alpha_r, \sigma_1^{-1}\beta_s\sigma_1] = [\beta_r, \sigma_1^{-1}\alpha_s\sigma_1] = [\beta_r, \sigma_1^{-1}\beta_s\sigma_1] = 1$ $(r < s)$;
  (SCR) $\sigma_1\beta_r\sigma_1\alpha_r\sigma_1 = \alpha_r\sigma_1\beta_r$.

Note that here composition of loops is written from the right.

The homology group $H_1(\Sigma) = H_1(\Sigma, \mathbb{Z})$ is equipped with its symplectic intersection form $\omega : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$, which counts intersection of curves with signs. The Heisenberg group of Σ is the central extension of $H_1(\Sigma)$ associated with the 2-cocycle ω. This means that the Heisenberg group $\mathcal{H}$ is defined by $\mathcal{H} \coloneqq \mathbb{Z} \times H_1(\Sigma)$ as a set with product

$$(k, x)(l, y) = (k + l + \omega(x, y), x + y). \tag{3}$$

Denote by $\psi \colon \mathcal{H} \twoheadrightarrow H_1(\Sigma)$ the projection morphism and by $i \colon \mathbb{Z} \to \mathcal{H}$ the inclusion. Then we have a short exact sequence

$$\{0\} \to \mathbb{Z} \xrightarrow{i} \mathcal{H} \xrightarrow{\psi} H_1(\Sigma) \to \{0\},$$

where the image of i is central in $\mathcal{H}$. The group $\mathcal{H}$ is generated by the central element $u = (1, 0)$ and $\tilde{x} = (0, x)$ with $x \in H_1(\Sigma)$. We may reduce the generating set by using a basis of $H_1(\Sigma)$. Let a_i, b_i, $1 \le i \le g$, be a symplectic basis of $H_1(\Sigma)$, with $\omega(a_i, b_j) = \delta_{ij}$. Then the group $\mathcal{H}$ is generated by $u = (1, 0)$ and $\tilde{a}_i = (0, a_i), \tilde{b}_i = (0, b_i)$, $1 \le i \le g$. The following proposition gives a presentation of $\mathcal{H}$.

Proposition 5. *The group $\mathcal{H}$ is generated by the elements $u, \tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_g, \tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_g$, where $\tilde{a}_i = (0, a_i)$ and $\tilde{b}_i = (0, b_i)$ for $1 \le i \le g$, with relations:*

$$\begin{aligned} u\tilde{a}_i &= \tilde{a}_i u; \\ u\tilde{b}_i &= \tilde{b}_i u; \\ \tilde{a}_i\tilde{b}_j &= \begin{cases} u^2\tilde{b}_j\tilde{a}_i & \text{if } i = j, \\ \tilde{b}_j\tilde{a}_i & \text{if } i \neq j. \end{cases} \end{aligned} \tag{4}$$

Proof. Let $\mathcal{H}'$ be the group with the above presentation and define a homomorphism $\mathcal{H}' \xrightarrow{\varsigma} \mathcal{H}$ by $u \mapsto (1, 0)$, $\tilde{a}_i \mapsto (0, a_i)$ and $\tilde{b}_i \mapsto (0, b_i)$. To check that this is well-defined, we verify the following relations, using the definition (3) of the product on $\mathcal{H}$.

- $\varsigma(u) = (1, 0)$ commutes with $\varsigma(\tilde{a}_i) = (0, a_i)$ and with $\varsigma(\tilde{b}_i) = (0, b_i)$,
- $\varsigma(\tilde{a}_i)$ commutes with $\varsigma(\tilde{b}_j)$ for $i \neq j$, since $\omega(a_i, b_j) = 0$,
- $\varsigma(\tilde{a}_i)\varsigma(\tilde{b}_i) = (\omega(a_i, b_i), a_i + b_i) = (1, a_i + b_i) = (2, 0)(-1, b_i + a_i) =$
  $= (2, 0)(\omega(b_i, a_i), b_i + a_i) = \varsigma(u)^2\varsigma(\tilde{b}_i)\varsigma(\tilde{a}_i)$.

Any $x \in \mathcal{H}'$ may be written in the form

$$x = u^k\tilde{a}_1^{l_1}\tilde{b}_1^{m_1} \dots \tilde{a}_g^{l_g}\tilde{b}_g^{m_g},$$

with $k, l_i, m_i \in \mathbb{Z}$, since we can reduce any word using the relations (4) if it is not already in this reduced form. For an element in this reduced form we have

$$\varsigma(x) = \left(k + \sum_i l_i m_i \ , \ \sum_i l_i a_i + m_i b_i\right), \tag{5}$$

from which it is immediate that ς is surjective. To verify injectivity, let $x \in \mathcal{H}'$, which we assume to be written in reduced form as above, and suppose that $\varsigma(x) = 0$. From the right-hand side of the formula (5), it follows that $l_i = m_i = 0$ for all $i \in \{1, \ldots, g\}$, and then from the left-hand side of (5) it follows that also $k = 0$, and hence x is trivial. Thus ς defines an isomorphism $\mathcal{H}' \cong \mathcal{H}$. ∎

From now on we identify $\mathcal{H}'$ and $\mathcal{H}$ via the isomorphism ς from the proof of Proposition 5, given by the formula (5). We state our main result for this section.

Theorem 6. *For $g \geq 1$ and $n \geq 2$, there exists a surjective homomorphism $\phi \colon \mathbb{B}_n(\Sigma_{g,1}) \twoheadrightarrow \mathcal{H}$.*

Proof. We define the homomorphism ϕ by $\sigma_i \mapsto u$, $\alpha_j \mapsto \tilde{a}_j$ and $\beta_j \mapsto \tilde{b}_j$ for $1 \leq i \leq n-1$ and $1 \leq j \leq g$. We show that this homomorphism is well-defined by checking the surface braid relations.

(**BR1**) $[\phi(\sigma_i), \phi(\sigma_j)] = [u, u] = 1$;

(**BR2**) $\phi(\sigma_i)\phi(\sigma_j)\phi(\sigma_i) = uuu = \phi(\sigma_j)\phi(\sigma_i)\phi(\sigma_j)$;

(**CR1**)
- $[\phi(\alpha_r), \phi(\sigma_i)] = [\tilde{a}_r, u] = \tilde{a}_r u \tilde{a}_r^{-1} u^{-1} = (0, a_r)(1, 0)(0, -a_r)(-1, 0) = (0, 0)$;
- $[\phi(\beta_r), \phi(\sigma_i)] = [\tilde{b}_r, u] = \tilde{b}_r u \tilde{b}_r^{-1} u^{-1} = (0, b_r)(1, 0)(0, -b_r)(-1, 0) = (0, 0)$;

(using the group law (3) on $\mathcal{H}$).

(**CR2**)
- $[\phi(\alpha_r), \phi(\sigma_1)\phi(\alpha_r)\phi(\sigma_1)] = [\tilde{a}_r, u\tilde{a}_r u]$;
- $[\phi(\beta_r), \phi(\sigma_1)\phi(\beta_r)\phi(\sigma_1)] = [\tilde{b}_r, u\tilde{b}_r u]$;

(using the group law (3) on $\mathcal{H}$ as for **CR1**).

(**CR3**)
- $[\phi(\alpha_r), \phi(\sigma_1^{-1})\phi(\alpha_s)\phi(\sigma_1)] = [\tilde{a}_r, u^{-1}\tilde{a}_s u]$;
- $[\phi(\alpha_r), \phi(\sigma_1^{-1})\phi(\beta_s)\phi(\sigma_1)] = [\tilde{a}_r, u^{-1}\tilde{b}_s u]$;
- $[\phi(\beta_r), \phi(\sigma_1^{-1})\phi(\alpha_s)\phi(\sigma_1)] = [\tilde{b}_r, u^{-1}\tilde{a}_s u]$;
- $[\phi(\beta_r), \phi(\sigma_1^{-1})\phi(\beta_s)\phi(\sigma_1)] = [\tilde{b}_r, u^{-1}\tilde{b}_s u]$;

(using the group law (3) on $\mathcal{H}$ as for **CR1**).

(**SCR**) $\phi(\sigma_1)\phi(\beta_r)\phi(\sigma_1)\phi(\alpha_r)\phi(\sigma_1) = u\tilde{b}_r u\tilde{a}_r u = u^2\tilde{a}_r u\tilde{b}_r$.

Thus ϕ is well-defined. Immediately from its definition we see that its image contains a set of generators of $\mathcal{H}$, so it is surjective. ∎

In the case $n \geq 3$, this quotient map has previously been considered in the articles [4, 5, 6], which also consider the more general setting where Σ is closed or has several boundary components. The alternative approach in those articles allows one to identify the kernel of ϕ as a characteristic subgroup. We include below a description of the kernel valid for $n \geq 2$.

Proposition 7. a) *For $n \geq 2$, the kernel of ϕ is the normal subgroup generated by the commutators $[\sigma_1, x]$ for $x \in \mathbb{B}_n(\Sigma_{g,1})$.*
b) *For $n \geq 3$, the kernel of ϕ is the subgroup of 3-commutators $\Gamma_3(\mathbb{B}_n(\Sigma_{g,1}))$.*

For the statement b), we refer to [4, Theorem 2]. More precisely, statement (10) on page 1416 of [4] is the analogous fact for the closed surface Σ_g: that there is a surjective homomorphism $\mathbb{B}_n(\Sigma_g) \twoheadrightarrow \mathcal{H}_g/\langle u^{2(n+g-1)}\rangle$ whose kernel is exactly $\Gamma_3(\mathbb{B}_n(\Sigma_g))$. The proof given there works also in our case where the surface has one boundary component and we do not quotient by $\langle u^{2(n+g-1)}\rangle$. In this paper we will use statement a) and focus on the case $n = 2$ in the explicit computations.

Proof. Let $K_n \subseteq \mathbb{B}_n(\Sigma_{g,1})$ be the normal subgroup generated by the commutators $[\sigma_1, x]$ for $x \in \mathbb{B}_n(\Sigma_{g,1})$. The image $\phi(\sigma_1)$ being central, we have $K_n \subseteq \ker(\phi)$, hence we get that ϕ can be factorised into a surjective homomorphism $\overline{\phi}\colon \mathbb{B}_n(\Sigma_{g,1})/K_n \to \mathcal{H}$. If we add centrality of σ_1 to the defining relations for $\mathbb{B}_n(\Sigma_{g,1})$, we may:

- replace (BR2) by $\sigma_i = \sigma_1$ for all i,
- remove (BR1), (CR1) and (CR2),
- replace (CR3) by commutators of all pairs of generators except for (α_r, β_r),
- replace (SCR) with $\alpha_r\beta_r = \sigma_1^2\beta_r\alpha_r$.

Finally the presentations of $\mathbb{B}_n(\Sigma_{g,1})/K_n$ and $\mathcal{H}$ coincide and $\overline{\phi}$ is an isomorphism, which proves a). ∎

In contrast to the case of $n \geq 3$, the kernel $\ker(\phi)$ when $n = 2$ lies strictly between the terms Γ_2 and Γ_3 of the lower central series of $\mathbb{B}_2(\Sigma_{g,1})$.

Proposition 8. *There are proper inclusions*

$$\Gamma_3(\mathbb{B}_2(\Sigma_{g,1})) \hookrightarrow \ker(\phi) \hookrightarrow \Gamma_2(\mathbb{B}_2(\Sigma_{g,1})).$$

Proof. By the above proposition, $\ker(\phi)$ is normally generated by commutators, so it must lie inside $\Gamma_2(\mathbb{B}_2(\Sigma_{g,1}))$. On the other hand, the Heisenberg group $\mathcal{H} = \mathcal{H}_g$ is a central extension of an abelian group, hence 2-nilpotent. The kernel of any homomorphism $G \to H$ with target a 2-nilpotent group contains $\Gamma_3(G)$, so $\ker(\phi)$ contains $\Gamma_3(\mathbb{B}_2(\Sigma_{g,1}))$. To see that $\ker(\phi)$ is not equal to Γ_2, it suffices to note that the Heisenberg group is not abelian. To see that $\ker(\phi)$ is not equal to Γ_3, we will construct a quotient

$$\psi\colon \mathbb{B}_2(\Sigma_{g,1}) \longrightarrow Q$$

where Q is 2-nilpotent and $[\sigma_1, a_1] \notin \ker(\psi)$. Given this for the moment, suppose for a contradiction that $\ker(\phi) = \Gamma_3$. Then we have $[\sigma_1, a_1] \in \ker(\phi) = \Gamma_3 \subseteq \ker(\psi)$, due to the fact that Q is 2-nilpotent, which is a contradiction.

It therefore remains to show that there exists a quotient Q with the claimed properties. In fact we will take $Q = D_4$, the dihedral group with 8 elements presented by $D_4 = \langle g, \tau \mid g^2 = \tau^2 = (g\tau)^4 = 1\rangle$. Let us set $\psi(a_i) = \psi(b_i) = g$ and $\psi(\sigma_1) = \tau$. It is easy to verify from the presentations that this is a well-defined surjective homomorphism. The dihedral group D_4 is 2-nilpotent (its centre is generated by $(g\tau)^2$ and the quotient by this element is isomorphic to the abelian group $(\mathbb{Z}/2)^2$), and we compute that $\psi([\sigma_1, a_1]) = (\tau g)^2 \neq 1$, which completes the proof. ∎

Remark 9. Although our construction in Theorem 6 of a quotient of $\mathbb{B}_n(\Sigma_{g,1})$ onto the Heisenberg group $\mathcal{H}$ uses the assumption that $n \geq 2$, there is also a quotient of this form for $n = 1$, arising slightly differently. By [23, §2], the quotient of $\mathbb{B}_1(\Sigma_{g,1}) = \pi_1(\Sigma_{g,1})$ by its subgroup $\Gamma_3(\pi_1(\Sigma_{g,1}))$ is the central extension of $H = H_1(\Sigma_{g,1})$ by $N = \wedge^2 H$ given by the 2-cocycle $(x, y) \mapsto x \wedge y \colon H \times H \to \wedge^2 H$. Composing this 2-cocycle with the quotient $\wedge^2 H \twoheadrightarrow \mathbb{Z}$ sending $a_i \wedge b_i$ and $-(b_i \wedge a_i)$ to 1 and sending every other element of the form $c \wedge d$ for $c, d \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ to 0, we obtain precisely the 2-cocycle $H \times H \to \mathbb{Z}$ defining the Heisenberg group $\mathcal{H}$ as a central extension. This induces a quotient $\pi_1(\Sigma_{g,1})/\Gamma_3 \twoheadrightarrow \mathcal{H}$ of central extensions of H. We therefore have a quotient

$$\mathbb{B}_1(\Sigma_{g,1}) = \pi_1(\Sigma_{g,1}) \longrightarrow\!\!\!\!\!\rightarrow \pi_1(\Sigma_{g,1})/\Gamma_3 \longrightarrow\!\!\!\!\!\rightarrow \mathcal{H}$$

whose kernel lies strictly between Γ_2 and Γ_3, as in the case $n = 2$.

2 Heisenberg homology

Using the homomorphism ϕ, any representation V of the Heisenberg group $\mathcal{H}$ becomes a module over $R = \mathbb{Z}[\mathbb{B}_n(\Sigma)]$. Following [21, Ch. 3.H] or [17, Ch. 5] we then have homology groups with local coefficients $H_*(\mathcal{C}_n(\Sigma); V)$. When V is the regular representation $\mathbb{Z}[\mathcal{H}]$, we simply write $H_*(\mathcal{C}_n(\Sigma); \mathcal{H})$. Let $\widetilde{\mathcal{C}}_n(\Sigma)$ be the regular covering of $\mathcal{C}_n(\Sigma)$ associated with the kernel of ϕ. Then $H_*(\mathcal{C}_n(\Sigma); \mathcal{H})$ is the homology of the singular chain complex $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))$ considered as a right $\mathbb{Z}[\mathcal{H}]$-module by deck transformations. Given a left representation V of $\mathcal{H}$, then $H_*(\mathcal{C}_n(\Sigma); V)$ is the homology of the complex $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \otimes_{\mathbb{Z}[\mathcal{H}]} V$.

Relative homology with local coefficients is defined in the usual way. We also use the *Borel-Moore* homology, defined by

$$H_n^{BM}(\mathcal{C}_n(\Sigma); V) = \varprojlim_T H_n(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma) \setminus T; V),$$

where the inverse limit is taken over all compact subsets $T \subset \mathcal{C}_n(\Sigma)$. In general, writing $\mathcal{K}(X)$ for the poset of compact subsets of a space X, the Borel-Moore homology module $H_n^{BM}(X, A; V)$ is the limit of the functor $H_n(X, A \cup (X \setminus -); V) \colon \mathcal{K}(X)^{\mathrm{op}} \to \mathrm{Mod}_R$ for any local system V on X and any subspace $A \subseteq X$. Under mild conditions, which are satisfied in our setting, the Borel-Moore homology is isomorphic to the homology of the chain complex of locally finite singular chains.

Borel-Moore homology is functorial with respect to proper maps. If $f \colon X \to Y$ is a proper map taking $A \subseteq X$ into $B \subseteq Y$, then there is an induced functor $f^{-1} \colon \mathcal{K}(Y) \to \mathcal{K}(X)$ by taking pre-images, and a natural transformation $H_n(X, A \cup (X \setminus -); f^*(V)) \circ f^{-1} \Rightarrow H_n(Y, B \cup (Y \setminus -); V)$ arising from the naturality of singular homology. Taking limits, we obtain

$$\begin{aligned} H_n^{BM}(X, A; f^*(V)) &= \lim H_n(X, A \cup (X \setminus -); f^*(V)) \\ &\longrightarrow \lim \left(H_n(X, A \cup (X \setminus -); f^*(V)) \circ f^{-1}\right) \\ &\longrightarrow \lim H_n(Y, B \cup (Y \setminus -); V) = H_n^{BM}(Y, B; V). \end{aligned}$$

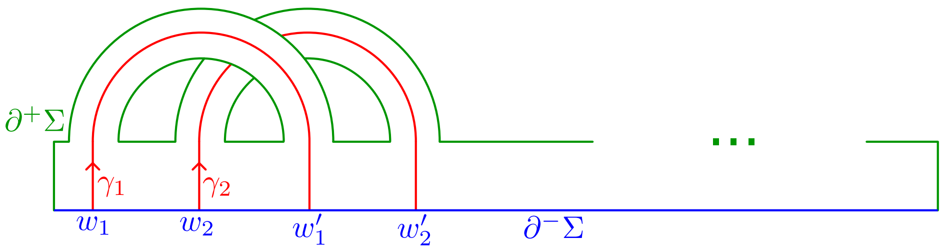
In particular, homeomorphisms are proper maps, so self-homeomorphisms of a space act on its Borel-Moore homology.

We will adapt a method used by Bigelow in the genus-0 case [10] (see also [1, 28, 2]) for computing the relative homology

$$H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V) = \varprojlim_T (\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup (\mathcal{C}_n(\Sigma) \setminus T); V),$$

where $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$ is the closed subspace of configurations containing at least one point in an interval $\partial^-(\Sigma) \subset \partial\Sigma$. In general for a pair (X, Y) the notation $\mathcal{C}_n(X, Y)$ will be used for configurations of n points in X containing at least one point in Y.

The surface Σ can be represented as a thickened interval $[0,1] \times I$ with $2g$ handles, attached as depicted below along $\{1\} \times W$, where W contains in this order the points $w_1, w_2, w_1', w_2', \ldots, w_{2g-1}, w_{2g}, w_{2g-1}', w_{2g}'$. We view Σ as a relative cobordism from $\partial^-(\Sigma) = \{0\} \times I$ to $\partial^+(\Sigma)$, where $\partial^+(\Sigma)$ is the closure of the complement of $\partial^-(\Sigma)$ in $\partial(\Sigma)$. For $1 \leq i \leq 2g$, γ_i denotes the union of the core of the i-th handle with $[0,1] \times \{w_i, w_i'\}$, oriented from w_i to w_i', and $\Gamma = \amalg_i \gamma_i$.



Let $\mathcal{K}$ be the set of sequences $k = (k_1, k_2, ..., k_{2g})$ such that k_i is a non-negative integer and $\sum_i k_i = n$. We will associate to each $k \in \mathcal{K}$ an element of the Borel-Moore relative homology $H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})$, as follows.

For $k \in \mathcal{K}$ we consider the submanifold $E_k \subset \mathcal{C}_n(\Sigma)$ consisting of all configurations having k_i points on γ_i. This manifold is *oriented* by the order of the points on each γ_i and increasing i. Moreover, it is a properly embedded half euclidean space $\mathbb{R}_+^n$ in $\mathcal{C}_n(\Sigma)$ with boundary in $\mathcal{C}_n(\Sigma, \partial^-(\Sigma))$. After connecting to the base point in $\mathcal{C}_n(\Sigma)$, E_k represents a homology class in $H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})$ which we also denote by E_k.

Theorem 10. *Let V be a representation of the Heisenberg group $\mathcal{H}$. Then, for $n \geq 2$, the module $H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$ is isomorphic to the direct sum $\bigoplus_{k \in \mathcal{K}} V$. Furthermore, it is the only non-vanishing module in $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$. In particular $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})$ is a free $\mathbb{Z}[\mathcal{H}]$-module of dimension $\binom{2g+n-1}{n}$ with basis $(E_k)_{k \in \mathcal{K}}$.*

Remark 11. Theorem 10 is true (with the same proof) more generally for Borel-Moore homology with coefficients in any representation V of the surface braid group $\mathcal{B}_n(\Sigma) = \pi_1(\mathcal{C}_n(\Sigma))$, not necessarily factoring through the quotient $\mathcal{B}_n(\Sigma) \twoheadrightarrow \mathcal{H}$. However, we will only need Theorem 10 for representations of the Heisenberg group.

Recall that a deformation retraction $h\colon [0,1]\times\Sigma \to \Sigma$ from Σ to $Y \subset \Sigma$ is a continuous map $(t,x) \mapsto h(t,x) = h_t(x)$ such that $h_0 = \mathrm{Id}_\Sigma$, $h_1(\Sigma) \subset Y$, and $(h_t)_{|Y} = \mathrm{Id}_Y$. We will prove the following lemma in Appendix A.

Lemma 12. *There exists a metric d on Σ inducing the standard topology and a deformation retraction h from Σ to $\Gamma \cup \partial^-(\Sigma)$, such that for all $0 \le t < 1$, the map h_t is a 1-Lipschitz embedding.*

Proof of Theorem 10. We use a metric d and a deformation retraction h from Lemma 12. For $\epsilon > 0$ and $Y \subset \Sigma$ we denote by $\mathcal{C}_n^\epsilon(Y)$ the subspace of configurations $x = \{x_1, x_2, ..., x_n\} \subset Y$ such that $d(x_i, x_j) < \epsilon$ for some $i \neq j$. If Y is closed, then $\mathcal{C}_n^\epsilon(Y)$ is a cofinal family of co-compact subsets of $\mathcal{C}_n(Y)$, which implies that for a pair (Y, Z) of closed subspaces of Σ, we have

$$H_*^{BM}(\mathcal{C}_n(Y), \mathcal{C}_n(Y,Z); V) \;\cong\; \lim_{0 \leftarrow \epsilon} H_*(\mathcal{C}_n(Y), \mathcal{C}_n(Y,Z) \cup \mathcal{C}_n^\epsilon(Y); V) \tag{6}$$

For $0 \le t \le 1$, let $\Sigma_t = h_t(\Sigma)$. For $t < 1$ we have an inclusion

$$(\mathcal{C}_n(\Sigma_t), \mathcal{C}_n(\Sigma_t, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_t) \;\subset\; (\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma))$$

which is a homotopy equivalence with homotopy inverse $\mathcal{C}_n(h_t)$, which is a map of pairs because h_t is 1-Lipschitz. So we have an inclusion isomorphism

$$H_*(\mathcal{C}_n(\Sigma_t), \mathcal{C}_n(\Sigma_t, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_t); V) \;\cong\; H_*(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma); V) \tag{7}$$

The compactness of Σ ensures that h_1 is the uniform limit of h_t, $t \to 1$, which implies that for $\epsilon > 0$ we may choose $t = t_\epsilon < 1$ such that for all $p \in \Sigma$ we have $d(h_t(p), h_1(p)) < \frac{\epsilon}{2}$. For such t, let $A_t \subset \mathcal{C}_n(\Sigma_t)$ be the subset of configurations $x = \{x_1, \dots, x_n\} \subset \Sigma_t$ such that $h_1(h_t^{-1}(x_i)) = h_1(h_t^{-1}(x_j))$ for some $i \neq j$. We have that A_t is closed and (by our definition of $t = t_\epsilon$) contained in the open set $\mathcal{C}_n^\epsilon(\Sigma_t)$. We therefore get an excision isomorphism

$$H_*(\mathcal{C}_n(\Sigma_t) \backslash A_t, (\mathcal{C}_n(\Sigma_t, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_t)) \backslash A_t; V) \;\cong\; H_*(\mathcal{C}_n(\Sigma_t), \mathcal{C}_n(\Sigma_t, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_t); V) \tag{8}$$

By applying $h_1 \circ (h_t)^{-1}$ on configurations, we obtain a well-defined map of pairs

$$(\mathcal{C}_n(\Sigma_t) \setminus A_t, (\mathcal{C}_n(\Sigma_t, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_t)) \setminus A_t) \;\longrightarrow\; (\mathcal{C}_n(\Sigma_1), \mathcal{C}_n(\Sigma_1, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_1))\ ,$$

which is a homotopy inverse to the inclusion. Here $\Sigma_1 = h_1(\Sigma)$ is equal to $\Gamma \cup \partial^-(\Sigma)$. By composing inclusions and excision maps, we obtain the inclusion isomorphism:

$$H_*(\mathcal{C}_n(\Sigma_1), \mathcal{C}_n(\Sigma_1, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_1); V) \;\cong\; H_*(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma); V) \tag{9}$$

Let $W^- = \{0\} \times W \subset \partial^-(\Sigma)$ and $U_\epsilon \subset \partial^-(\Sigma)$ be defined by the condition $x \in U_\epsilon \Leftrightarrow d(x, W^-) < \frac{\epsilon}{2}$, and $\Gamma_\epsilon = \Gamma \cup U_\epsilon$. In the above left hand side group, we may apply excision with the closed subset $\mathcal{C}_n(\Sigma_1, \partial^-(\Sigma) \setminus U_\epsilon)$, which gives

$$H_*(\mathcal{C}_n(\Gamma_\epsilon), \mathcal{C}_n(\Gamma_\epsilon, U_\epsilon) \cup \mathcal{C}_n^\epsilon(\Gamma_\epsilon); V) \;\cong\; H_*(\mathcal{C}_n(\Sigma_1), \mathcal{C}_n(\Sigma_1, \partial^-(\Sigma)) \cup \mathcal{C}_n^\epsilon(\Sigma_1); V) \tag{10}$$

We will end with one more excision removing configurations which contain 2 points in the same component of U_ϵ followed by a deformation retraction to configurations in Γ and finally obtain

$$H_*(\mathcal{C}_n(\Gamma), \mathcal{C}_n(\Gamma, W^-)) \cup \mathcal{C}_n^\epsilon(\Gamma); V) \;\cong\; H_*(\mathcal{C}_n(\Gamma_\epsilon), \mathcal{C}_n(\Gamma_\epsilon, U_\epsilon) \cup \mathcal{C}_n^\epsilon(\Gamma_\epsilon); V) \tag{11}$$

Taking the limit $0 \leftarrow \epsilon$ in the composition of the isomorphisms from equations eqs. (7) to (11), we obtain

$$H_*^{BM}(\mathcal{C}_n(\Gamma), \mathcal{C}_n(\Gamma, W^-)); V) \;\cong\; H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V) \tag{12}$$

Now we observe that $(\mathcal{C}_n(\Gamma), \mathcal{C}_n(\Gamma, W^-)$ is the disjoint union of the relative cells $(E_k, \partial(E_k))$ for $k \in \mathcal{K}$. It follows that the Borel-Moore homology (12) is trivial when $* \neq n$, and that the Borel-Moore homology classes E_k represent a basis over $\mathbb{Z}[\mathcal{H}]$ when $* = n$, which achieves the proof of Theorem 10. ■

3 Action of mapping classes

The *mapping class group* of Σ, denoted by $\mathfrak{M}(\Sigma)$, is the group of orientation preserving diffeomorphisms of Σ fixing the boundary pointwise, modulo isotopies relative to the boundary. The isotopy class of a diffeomorphism f is denoted by $[f]$. An oriented self-diffeomorphism fixing the boundary pointwise $f \colon \Sigma \to \Sigma$ gives us a homeomorphism $\mathcal{C}_n(f) \colon \mathcal{C}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$, defined by $\{x_1, x_2, \dots, x_n\} \mapsto \{f(x_1), f(x_2), \dots, f(x_n)\}$. If we ensure that the basepoint configuration of $\mathcal{C}_n(\Sigma)$ is contained in $\partial\Sigma$, then it is fixed by $\mathcal{C}_n(f)$ and this in turn induces a homomorphism $f_{\mathbb{B}_n(\Sigma)} = \pi_1(\mathcal{C}_n(f)) \colon \mathbb{B}_n(\Sigma) \to \mathbb{B}_n(\Sigma)$, which depends only on the isotopy class $[f]$ of f.

3.1 Action on the Heisenberg group

We first study the induced action on the Heisenberg group quotient.

Proposition 13. *There exists a unique homomorphism $f_{\mathcal{H}} \colon \mathcal{H} \to \mathcal{H}$ such that the following square commutes:*

$$\begin{array}{ccc} \mathbb{B}_n(\Sigma) & \xrightarrow{f_{\mathbb{B}_n(\Sigma)}} & \mathbb{B}_n(\Sigma) \\ \phi \downarrow & & \downarrow \phi \\ \mathcal{H} & \xrightarrow{f_{\mathcal{H}}} & \mathcal{H} \end{array} \tag{13}$$

Thus, there is an action of $\mathfrak{M}(\Sigma)$ on the Heisenberg group $\mathcal{H}$ given by

$$\Psi \colon f \mapsto f_{\mathcal{H}} \colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}(\mathcal{H}). \tag{14}$$

Proof. Since ϕ is surjective, the homomorphism $f_{\mathcal{H}}$ will be uniquely determined by the formula $f_{\mathcal{H}}(\phi(\gamma)) = \phi(f_{\mathbb{B}_n(\Sigma)}(\gamma))$ if it exists. To show that it exists, we need to show that the composition $\phi \circ f_{\mathbb{B}_n(\Sigma)}$ factorises through ϕ, which is equivalent to saying that $f_{\mathbb{B}_n(\Sigma)}$ sends $\ker(\phi)$ into itself.

The braid σ_1 is supported in a sub-disc $D \subset \Sigma$ containing the base configuration. Let $T \subset \Sigma$ be a tubular neighbourhood of $\partial\Sigma$ containing D. Since f fixes $\partial\Sigma$ pointwise, we may isotope f so that it is the identity on T, in particular on D, which implies that $f_{\mathbb{B}_n(\Sigma)}$ fixes σ_1. We then deduce from part (a) of Proposition 7 that $f_{\mathbb{B}_n(\Sigma)}$ sends $\ker(\phi)$ to itself, which completes the proof. ∎

3.2 Structure of automorphisms of the Heisenberg group.

Recall that the centre of the Heisenberg group $\mathcal{H}$ is infinite cyclic, generated by the element u. Any automorphism of $\mathcal{H}$ must therefore send u to $u^{\pm 1}$.

Definition 14. We denote the index-2 subgroup of those automorphisms of $\mathcal{H}$ that fix u by $\mathrm{Aut}^+(\mathcal{H})$, and call these *orientation-preserving*.

From the proof of Proposition 13, we observe that, for any $f \in \mathfrak{M}(\Sigma)$, the automorphism $f_{\mathcal{H}}$ is orientation-preserving in the sense of Definition 14. We may therefore refine the action Ψ as follows:

$$\Psi\colon f \mapsto f_{\mathcal{H}}\colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}^+(\mathcal{H}). \tag{15}$$

The quotient of $\mathcal{H}$ by its centre may be canonically identified with $H = H_1(\Sigma)$, so every automorphism of $\mathcal{H}$ induces an automorphism of H. Moreover, if it is orientation-preserving, the induced automorphism of H preserves the symplectic structure, so we have a homomorphism $\mathrm{Aut}^+(\mathcal{H}) \to Sp(H)$ denoted $\varphi \mapsto \overline{\varphi}$. In addition, there is a function $\mathrm{Aut}^+(\mathcal{H}) \to H^* = \mathrm{Hom}(H,\mathbb{Z})$ defined by sending φ to $\varphi^\diamond = \mathrm{pr}_1(\varphi(0,-))$, where we are using the description $\mathcal{H} = \mathbb{Z} \times H$. The fact that $\varphi^\diamond$ is really a *homomorphism* $H \to \mathbb{Z}$ uses the fact that the automorphism of H induced by φ preserves the symplectic structure.

Lemma 15. *The homomorphism* $\mathrm{Aut}^+(\mathcal{H}) \to Sp(H)$ *and the function* $\mathrm{Aut}^+(\mathcal{H}) \to H^*$ *induce an isomorphism*

$$\mathrm{Aut}^+(\mathcal{H}) \cong H^* \rtimes Sp(H), \quad \varphi \mapsto (\varphi^\diamond, \overline{\varphi}) \tag{16}$$

where the semi-direct product structure on the right-hand side is induced by the natural action of $Sp(H)$ *on* H^*.

Proof. This is proven in Appendix B. ∎

Remark 16. Fixing a symplectic basis of H, the right-hand side of (16) is a subgroup of $\mathbb{Z}^{2g} \rtimes GL_{2g}(\mathbb{Z})$, which may be embedded into $GL_{2g+1}(\mathbb{Z})$. In this way, any orientation-preserving action of a group G on $\mathcal{H}$ may be viewed as a linear representation of G over $\mathbb{Z}$ of rank $2g+1$.

Lemma 15 asserts that the general form of an oriented automorphism φ is

$$\varphi(k,x) = (k + \varphi^\diamond(x), \overline{\varphi}(x)) \ ,$$

where $\varphi^\diamond \in H^*$ and $\overline{\varphi} \in Sp(H)$ is the induced symplectic automorphism. From formula (14) we observe that, for any $f \in \mathfrak{M}(\Sigma)$, the automorphism $f_{\mathcal{H}}$ is orientation-preserving in the sense of Definition 14. Hence for a mapping class $f \in \mathfrak{M}(\Sigma)$, the map $f_{\mathcal{H}}$ is represented as follows

$$f_{\mathcal{H}} : (k, x) \mapsto (k + \delta_f(x), f_*(x)) \tag{17}$$

where $\delta_f = f_{\mathcal{H}}^\diamond \in H^* = H^1(\Sigma)$.

3.3 Recovering Morita's crossed homomorphism.

We recall briefly the notion of a *crossed homomorphism*. Let G be a group acting on an abelian group K.

Definition 17. A *crossed homomorphism* $\theta\colon G \to K$ is a function with the property that $\theta(g_1 g_2) = \theta(g_1) + g_1\theta(g_2)$ for all $g_1, g_2 \in G$. A *principal crossed homomorphism* is one of the form $g \mapsto gh - h$ for a fixed element $h \in K$. Notice that every principal crossed homomorphism restricts to zero on the kernel of the action of G on K.

Remark 18. Crossed homomorphisms $G \to K$ are in one-to-one correspondence with lifts

$$\begin{array}{ccc} G & \dashrightarrow & K \rtimes \mathrm{Aut}(K) \\ & \searrow & \downarrow \\ & & \mathrm{Aut}(K), \end{array}$$

where the diagonal arrow is the given action of G on K. Often, we will have $K = H^*$ for a free abelian group H, with the G-action on K induced from one on H. In this case there is a natural (anti-)isomorphism $\mathrm{Aut}(H) \cong \mathrm{Aut}(K)$, and under this identification crossed homomorphisms $G \to H^*$ are in one-to-one correspondence with homomorphisms $G \to H^* \rtimes \mathrm{Aut}(H)$ lifting the action $G \to \mathrm{Aut}(H)$ of G on H.

Notation 19. The crossed homomorphism $G \to H^*$ corresponding to a homomorphism $\Theta\colon G \to H^* \rtimes \mathrm{Aut}(H)$ will be denoted by $\Theta^\diamond$.

Remark 20. Crossed homomorphisms form an abelian group under pointwise addition, and principal crossed homomorphisms form a subgroup. The quotient may be identified with the first cohomology group $H^1(G; K)$.

We will need the following lemma later on.

Lemma 21. *Let G be a group acting on an abelian group K, and denote by $N \subseteq G$ the kernel of this action. Let $S \subseteq N$ be a subset such that*

$$T = \{gsg^{-1} \mid s \in S, g \in G\} \subseteq N$$

generates N. If two crossed homomorphisms $\theta_1, \theta_2\colon G \to K$ agree on S, then they agree on N.

Note that we do *not* assume that S normally generates N; we assume only that N is generated by S together with all of its conjugates by elements of the larger group G.
Proof. Since T generates N, it will suffice to show that θ_1 and θ_2 agree on T. Let $s \in S$ and $g \in G$. We know by hypothesis that $\theta_1(s) = \theta_2(s)$, and we need to show that $\theta_1(gsg^{-1}) = \theta_2(gsg^{-1})$. First, observe that, for $i = 1, 2$, we have

$$\theta_i(g) + g.\theta_i(g^{-1}) = \theta_i(gg^{-1}) = \theta_i(1) = 0.$$

Using this, and the fact that $s \in N$, so it acts trivially on K, we deduce that

$$\begin{aligned} \theta_i(gsg^{-1}) &= \theta_i(g) + g.\theta_i(s) + gs.\theta_i(g^{-1}) \\ &= \theta_i(g) + g.\theta_i(s) + g.\theta_i(g^{-1}) \\ &= g.\theta_i(s). \end{aligned}$$

Thus $\theta_1(gsg^{-1}) = g.\theta_1(s) = g.\theta_2(s) = \theta_2(gsg^{-1})$, as required. ■

In [29], Morita introduced a crossed homomorphism $\mathfrak{M}(\Sigma) \to H^1(\Sigma)$, $f \mapsto \mathfrak{d}_f$ representing a generator for $H^1(\mathfrak{M}(\Sigma), H^1(\Sigma)) \cong \mathbb{Z}$. (cf. Proposition 27). We will recover this crossed homomorphism from the action $f \mapsto f_{\mathcal{H}}$ on the Heisenberg group.

Proposition 22. *The map $\delta\colon \mathfrak{M}(\Sigma) \to H^1(\Sigma)$, $f \mapsto \delta_f$, is a crossed homomorphism equal to Morita's $\mathfrak{d}$.*

Proof. We first show that δ is a crossed homomorphism. Let f, g be mapping classes, then we have for $(k, x) \in \mathcal{H}$

$$(g \circ f)_{\mathcal{H}}(k, x) = g_{\mathcal{H}}(k + \delta_f(x), f_*(x)) = (k + \delta_f(x) + \delta_g(f_*(x), (g \circ f)_*(x)) \; .$$

We do get $\delta_{g \circ f}(x) = \delta_f(x) + f^*(\delta_g)(x)$.

The formula for the Morita crossed homomorphism a priori depends on a choice of free generators of $\pi_1(\Sigma)$ while δ does not. We will use as generators the loops given by the first strand in the generators α_i, β_i of the braid group $\mathbb{B}_n(\Sigma)$, and keep the same notation. For $\gamma \in \pi_1(\Sigma)$, let us denote by γ_i the element in the free group generated by α_i, β_i that is the image of γ under the homomorphism which maps the other generators to 1. Then we have a decomposition

$$\gamma_i = \alpha_i^{\nu_1} \beta_i^{\mu_1} \dots \alpha_i^{\nu_m} \beta_i^{\mu_m} \; ,$$

where ν_j and μ_j are 0, -1 or 1. The integer $d_i(\gamma)$ is then defined[1] by

$$\begin{aligned} d_i(\gamma) &= \sum_{j=1}^{m} \nu_j \sum_{k=j}^{m} \mu_k - \sum_{j=1}^{m} \mu_j \sum_{k=j+1}^{m} \nu_k \\ &= \sum_{j=1}^{m} \sum_{k=1}^{m} \iota_{jk} \nu_j \mu_k, \end{aligned}$$

[1] There is a small misprint in [29].

where $\iota_{jk} = +1$ when $j \leq k$ and $\iota_{jk} = -1$ when $j > k$. The definition for the Morita crossed homomorphism is as follows:

$$\mathfrak{d}_f([\gamma]) = \sum_{i=1}^{g} d_i(f_\sharp(\gamma)) - d_i(\gamma) \ .$$

If $\gamma \in \pi_1(\Sigma)$ is the first strand of a pure braid also denoted γ, then the above decomposition of γ used for the definition of d_i is also a decomposition in the generators of the braid group, and from the definition of the product in $\mathcal{H}$ we have that

$$\phi(\gamma) = \left(\sum_{i=1}^{g} d_i(\gamma), [\gamma] \right) \in \mathcal{H} \ .$$

This can be checked by recursion on the length of γ. It can also be deduced from [29, Lemma 6.1]. The equality $\mathfrak{d}_f = \delta_f$ follows. ■

3.4 Action of the Torelli subgroup.

Recall that the *Torelli subgroup* $\mathfrak{T}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$ consists of those elements of the mapping class group whose natural action on $H_1(\Sigma)$ is trivial. The restriction of the crossed homomorphism $\delta : f \mapsto \delta_f$ on the Torelli group is an homomorphism. We will first describe this homomorphism in relation with the action of the Torelli group on homotopy classes of vector fields. Recall that the set of homotopy classes of non singular vectors fields $\Xi(\Sigma)$ support a natural simply transitive action of $H^1(\Sigma)$ (affine structure), and the action of $\mathfrak{M}(\Sigma)$ is compatible with this action. It follows that the Torelli group acts by translation on $\Xi(\Sigma)$, which defines an homomorphism $e : \mathfrak{T}(\Sigma) \to H^1(\Sigma)$. A formula for $e(f)([\gamma])$ where γ is a regular curve is the variation of the winding number. For convenience we recall about winding number below.

Fix a Riemannian metric. A non-vanishing vector field X gives a trivialisation of the unit tangent bundle $T_1(\Sigma) \cong \Sigma \times S^1$. The winding number $\omega_X(\gamma)$ of a regular oriented curve γ is the degree of the second component of the unit tangent vector. It can be computed as follows. Assuming that γ is transverse to X except at a finite set $\gamma \pitchfork X$ of points, where it looks locally as in Figure 1, then

$$\omega_X(\gamma) = \sum_{p \in \gamma \pitchfork X} \mathrm{sgn}(p),$$

where $\mathrm{sgn}(p)$ is defined in Figure 1.

The Chillingwoth homomorphism e, studied in [15, 23], is defined by

$$e_X(f)([\gamma]) = \omega_X(f \circ \gamma) - \omega_X(\gamma) \ .$$

Its kernel is the *Chillingworth subgroup*. Note that e does not depend on X, but extends to a crossed homomorphism $e_X : \mathfrak{M}(\Sigma) \to H^1(\Sigma)$ which does.

The following lemma is Proposition 3.7 of [13]. The proof there uses [30]. We give an independent proof below.

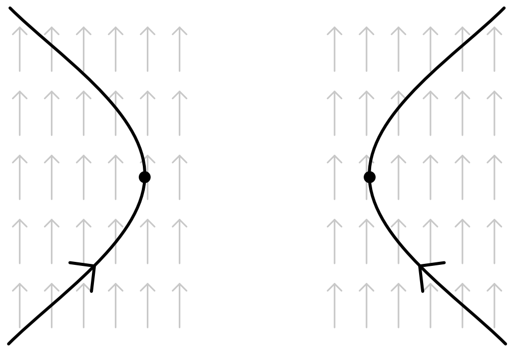


Figure 1: The sign of a point on γ that is tangent to X.

Lemma 23. *The homomorphisms δ and e coincide on the Torelli group and have image $\delta(\mathfrak{T}(\Sigma)) = 2H^1(\Sigma)$.*

From the formula (17) we get that the kernel of the action Ψ is included in the Torelli group so that we get this kernel as a corollary of Lemma 23.

Proposition 24. *For any genus $g \geq 1$, we have $\ker(\Psi) = \mathrm{Chill}(\Sigma)$.*

Proof. As $\ker(\Psi) \subseteq \mathfrak{T}(\Sigma)$, we may restrict to the Torelli group, at which point we see from formula (17) and Lemma 23 that $\ker(\Psi) = \ker(\delta) = \ker(e) = \mathrm{Chill}(\Sigma)$. ■

Denote by $\mathrm{Inn}(\mathcal{H})$ the group of inner automorphisms of the Heisenberg group $\mathcal{H}$. From Lemma 23, we also deduce the following.

Proposition 25. *For any genus $g \geq 1$, we have $\Psi^{-1}(\mathrm{Inn}(\mathcal{H})) = \mathfrak{T}(\Sigma)$.*

Proof. Conjugation in the Heisenberg group $\mathcal{H}$ is given by the formula

$$(l, x)(k, y)(-l, -x) = (l, y)(k, x)(-l, -x) = (k + 2\omega(x, y), y) \tag{18}$$

First, if $f_{\mathcal{H}}$ acts by inner automorphisms, then its induced action on H must be trivial. This means that f lies in the Torelli group. Conversely, if $f \in \mathfrak{T}(\Sigma)$, we have from Lemma 23 that δ_f is in $2H^1(\Sigma)$. Using Poincaré duality, we obtain $x \in H$ such that $\delta(y) = 2\omega(x, y)$ for every y. With the formula (18) we get that $f_{\mathcal{H}}$ is inner. ■

Proof of Lemma 23. The Torelli group is generated by genus one bounding pairs [22, Theorem 2], and this generating set is a single conjugacy class in the full mapping class group. By Lemma 21 and the fact that both δ and e are crossed homomorphisms defined on the full mapping class group, it will suffice to show that they agree on one particular genus one bounding pair, and take values in $2H^1(\Sigma)$ on this element. Specifically, we will take this element to be

$$f = BP(\gamma, \delta) = T_\gamma . T_\delta^{-1},$$

the genus one bounding pair diffeomorphism depicted in Figure 2, and we will show that both elements $e(f)$ and δ_f of $H^1(\Sigma) \cong \mathrm{Hom}(H_1(\Sigma), \mathbb{Z})$ are equal to the homomorphism $H_1(\Sigma) \to \mathbb{Z}$ given by

$$a_1 \mapsto 2 \quad , \quad a_i \mapsto 0 \text{ for } i \geq 2 \quad \text{and} \quad b_i \mapsto 0 \text{ for } i \geq 1. \tag{19}$$

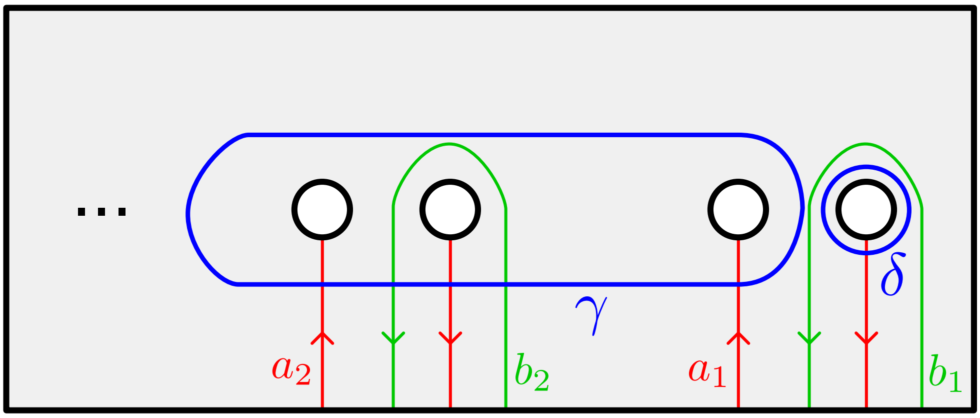


Figure 2: The surface Σ is obtained by identifying the $2g$ interior boundary components (4 depicted above) in g pairs by reflections. The bounding pair map from the proof of Lemma 23 is $BP(\gamma, \delta) = T_\gamma . T_\delta^{-1}$, for the blue curves γ and δ. The red and green arcs form a symplectic basis for the first homology of Σ relative to the bottom edge $\partial^- \Sigma$.

We first calculate δ_f from the automorphism $f_{\mathcal{H}}$. We may directly read off from Figure 2 the effect of f on the elements a_i and b_i of $\mathcal{H}$. It clearly acts trivially except possibly on the three elements $\tilde{a}_2 = (0, a_2)$, $\tilde{b}_2 = (0, b_2)$ and $\tilde{a}_1 = (0, a_1)$, since the others may be realised disjointly from $\gamma \cup \delta$, and:

$$\begin{aligned}
\tilde{a}_1 &\mapsto [\tilde{a}_2, \tilde{b}_2].\tilde{a}_1 = u^2 \tilde{a}_1 = (2, a_1) \\
\tilde{a}_2 &\mapsto [\tilde{a}_2, \tilde{b}_2].\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1}.\tilde{a}_2.\tilde{a}_1 \tilde{b}_1^{-1} \tilde{a}_1^{-1}.[\tilde{a}_2, \tilde{b}_2]^{-1} = \tilde{a}_2 \\
\tilde{b}_2 &\mapsto [\tilde{a}_2, \tilde{b}_2].\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1}.\tilde{b}_2.\tilde{a}_1 \tilde{b}_1^{-1} \tilde{a}_1^{-1}.[\tilde{a}_2, \tilde{b}_2]^{-1} = \tilde{b}_2.
\end{aligned}$$

This gives (19) for δ_f.

To calculate $e(f)$, we use the alternative model for the surface Σ, the bounding pair (γ, δ) and the symplectic basis a_i, b_i for H depicted in Figure 3. This model for Σ has the advantage of having an obvious unit vector field X, which simply points *upwards* according to the standard framing of the page.

Using this vector field X and comparing to Figure 1, we observe that the winding numbers of the symplectic generators a_i and b_i (more precisely, their smooth, closed representatives pictured in Figure 3) are given by

$$\omega_X(a_i) = -1 \qquad \text{and} \qquad \omega_X(b_i) = +1.$$

We recall that, by definition, $e_X(f)(c) = \omega_X(f \circ \bar{c}) - \omega_X(\bar{c}) \in \mathbb{Z}$ for any $c = [\bar{c}] \in H$. We clearly have $f \circ \bar{c} = \bar{c}$ for $\bar{c} = a_i$ or b_i with $i \geq 3$ or for $\bar{c} = b_1$, since these curves may be

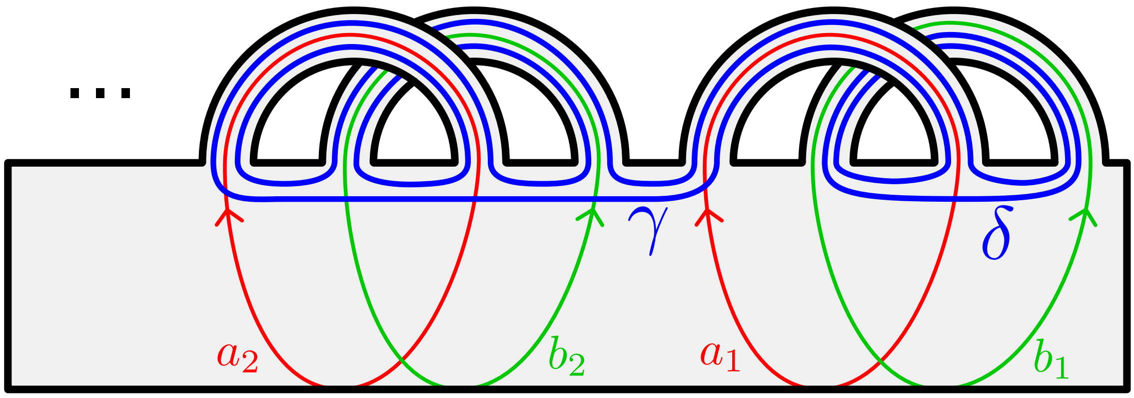


Figure 3: An alternative model for the surface Σ, the bounding pair (γ, δ) and the symplectic basis for the first homology of Σ relative to the bottom edge $\partial^- \Sigma$.

represented disjointly from $\gamma \cup \delta$. Hence $e_x(f)([\bar{c}]) = 0$ for these $\bar{c}$. The curve $f \circ a_1$ is depicted in Figure 4.

There are precisely three points on this curve where its tangent vector is equal to the vector field X, i.e., where its tangent vector is pointing vertically upwards: two are positive and one is negative (compare the local models in Figure 1), hence

$$\begin{aligned} e_X(f)(a_1) &= \omega_X(f \circ a_1) - \omega_X(a_1) \\ &= (2-1) - (-1) \\ &= 2. \end{aligned}$$

Now let $\bar{c}$ be either a_2 or b_2. In this case the effect of f is simply to conjugate $\bar{c}$ by γ, so we have that

$$\begin{aligned} \omega_X(f \circ \bar{c}) &= \omega_X(\gamma) + \omega_X(\bar{c}) - \omega_X(\gamma) \\ &= \omega_X(\bar{c}), \end{aligned}$$

since positive/negative tangencies with X for γ are negative/positive tangencies with X for γ^{-1} respectively, and so $e_X(f)([\bar{c}]) = \omega_X(f \circ \bar{c}) - \omega_X(\bar{c}) = 0$. Thus we have shown that the homomorphism $e(f) \colon H \to \mathbb{Z}$ is given by (19). ■

3.5 The Trapp representation.

We next recall the *Trapp representation* [37], and show that our representation of $\mathfrak{M}(\Sigma)$ on $\mathcal{H}$ may be identified with it (up to "coboundaries") when the genus of Σ is at least 2. This recovers Proposition 24, since the kernel of the Trapp representation is precisely the Chillingworth subgroup $\mathrm{Chill}(\Sigma)$ under this condition [37, Corollary 2.7].

Definition 26. The representation of Trapp [37] is defined as a homomorphism

$$\Phi_X \colon \mathfrak{M}(\Sigma) \longrightarrow H^* \rtimes Sp(H) \subset GL_{2g+1}(\mathbb{Z}) \tag{20}$$

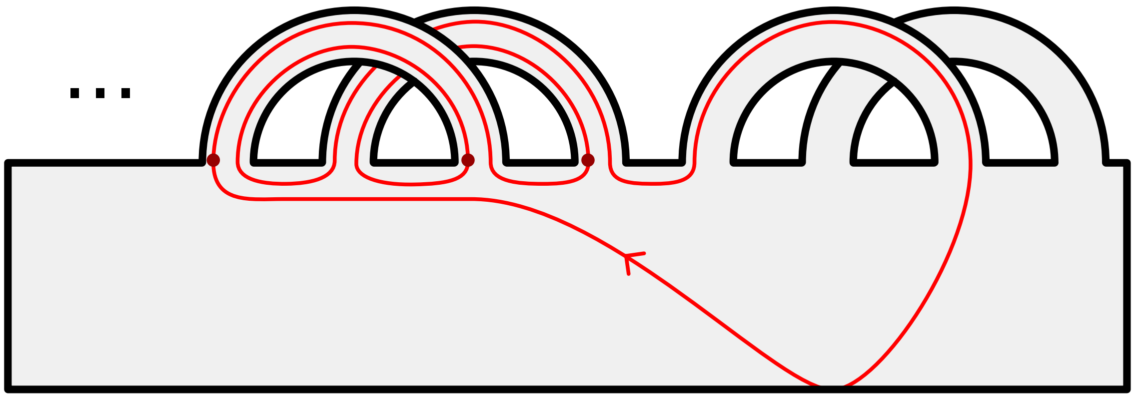


Figure 4: The curve $f \circ a_1$ for $f = T_\gamma.T_\delta^{-1}$. The three points where its tangent vector points vertically upwards are marked with dark red points: the left-most one is negative according to Figure 1, and the other two are positive.
[At first sight it may look like there are two more, but these are not allowed since they do not fit either of the local models of Figure 1. We therefore perturb the curve slightly to get rid of these two tangencies with the vector field X. Alternatively, we may perturb it differently, to turn each of these disallowed tangencies into a pair of two allowed tangencies with opposite signs, which will therefore cancel in the expression for $\omega_X(f \circ a_1)$.]

(*cf.* Remark 16), lifting the symplectic action $\mathfrak{M}(\Sigma) \to Sp(H)$. Viewed as a homomorphism into $H^* \rtimes \mathrm{Aut}(H)$, it therefore corresponds by Remark 18 to a crossed homomorphism $\mathfrak{M}(\Sigma) \to H^*$. This crossed homomorphism is the variation of the winding number with respect to a fixed non singular vector field X on Σ.

We now wish to compare the two homomorphisms

$$\Phi_X = (20) \quad \text{and} \quad \Psi = (16) \circ (15) \quad : \quad \mathfrak{M}(\Sigma) \longrightarrow H^* \rtimes \mathrm{Aut}(H)$$

corresponding, respectively, to the crossed homomorphisms

$$\Phi_X^\diamond = e_X \quad \text{and} \quad \Psi^\diamond = \delta \quad : \quad \mathfrak{M}(\Sigma) \longrightarrow H^*.$$

Proposition 27. *For $g \geq 2$, the crossed homomorphisms e_X and δ represent the same cohomology class in $H^1(\mathfrak{M}(\Sigma); H^*) \cong \mathbb{Z}$. In other words, they are equal modulo principal crossed homomorphisms.*

Proof of Proposition 27. We will use the homomorphism

$$H^1(\mathfrak{M}(\Sigma); H^*) \longrightarrow \mathrm{Hom}(\mathfrak{T}(\Sigma), H^*) \tag{21}$$

given by restricting a crossed homomorphism $\mathfrak{M}(\Sigma) \to H^*$ to the Torelli group. This is well-defined since, as mentioned before, principal crossed homomorphisms are trivial on the Torelli group. The right-hand side of (21) is rather large: in fact, by a theorem of Johnson [24], the abelianisation of $\mathfrak{T}(\Sigma)$ is isomorphic to $\wedge^3 H \oplus$ (torsion), so

$\mathrm{Hom}(\mathfrak{T}(\Sigma), H^*) \cong \mathrm{Hom}(\wedge^3 H, H^*)$, which is free abelian of rank $2g\binom{2g}{3}$. However, it has the advantage that it is easier to detect when elements are equal, since it is just a group of homomorphisms (rather than crossed homomorphisms modulo principal ones). On the other hand, the left-hand side of (21) is much smaller. Indeed, Morita proved in [29, Proposition 6.4] that the group $H^1(\mathfrak{M}(\Sigma); H^*)$ is infinite cyclic. (In fact, it is generated by $[\mathfrak{d}]$, which we know by Proposition 22 is equal to $[\delta]$, but we will not need this.) In Lemma 23 we have proven that δ and e_X coincide (and are non-trivial) on the Torelli subgroup. Since $\mathrm{Hom}(\mathfrak{T}(\Sigma), H^*)$ is torsion-free, the homomorphism (21) is injective and the result follows. ∎

Remark 28. In summary, we have considered three different crossed homomorphisms

$$\delta, \mathfrak{d}, e_X \colon \mathfrak{M}(\Sigma) \longrightarrow H^1(\Sigma),$$

where δ is the crossed homomorphism corresponding to the natural action Ψ of the mapping class group on the Heisenberg group, $\mathfrak{d}$ is Morita's combinatorially-defined crossed homomorphism and e_X is the Chillingworth crossed homomorphism (depending on a choice of non-vanishing vector field X on Σ). We have shown (Proposition 22) that $\delta = \mathfrak{d}$ on $\mathfrak{M}(\Sigma)$ and (Lemma 23) that $\delta = e_X$ when restricted to $\mathfrak{T}(\Sigma)$. In Proposition 27 just above, we used the latter fact to deduce the stronger statement that $\delta = e_X$ on $\mathfrak{M}(\Sigma)$ modulo principal crossed homomorphisms.

However, we note that only the weaker statement of Lemma 23 was needed to deduce (Proposition 24) that $\ker(\Psi) = \mathrm{Chill}(\Sigma)$. An alternative argument to prove this would have been to apply Morita's result [29, Proposition 6.4] that $H^1(\mathfrak{M}(\Sigma); H^*)$ is infinite cyclic and observe that $[\delta]$ and $[e_X]$ are both non-zero, and thus $k.\delta = \ell.e_X$ on the Torelli group, for non-zero integers k, ℓ, which implies that the kernels of δ and e_X coincide on the Torelli group. (This argument is outlined in [33, §5.4.1], for example.) However, we have preferred here to give a direct proof that $\delta = e_X$ on the Torelli group (so $k = \ell = 1$), avoiding the need to use the above result of Morita.

4 Twisted actions and central extensions

4.1 Untwisting.

In order to apply the construction of the previous section to define a representation of (a central extension of) the Torelli group of Σ (rather than just the smaller Chillingworth subgroup), we will need a certain "untwisting" trick. This may either be done explicitly at the level of chain complexes, or it may be done already at the level of spaces equipped with local systems, *before* passing to chains. In this section, we explain the latter point of view. We first describe a general algebraic trick for "untwisting" representations of groups, and then augment it to a fibrewise version, which may be applied to local systems on spaces.

Let G, K be groups and let M be a right $\mathbb{Z}[K]$-module. Suppose that G has a *twisted* left action on M, in the sense that there are actions

$$\alpha\colon G \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(M)$$
$$\beta\colon G \longrightarrow \mathrm{Aut}(K)$$

such that $\alpha(g)(m.h) = \alpha(g)(m).\beta(g)(h)$ for all $g \in G$, $h \in K$ and $m \in M$. Moreover, suppose that the action β of G on K is by *inner* automorphisms, and define a central extension of G by the following pullback (the symbol $\lrcorner$ denotes a pullback square):

$$\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
\mathcal{Z}(K) & \xrightarrow{\ \mathrm{id}\ } & \mathcal{Z}(K) \\
\downarrow & & \downarrow \\
\widetilde{G} & \xrightarrow{\ \theta\ } & K \\
{\scriptstyle \pi}\downarrow & \lrcorner & \downarrow \\
G & \xrightarrow{\ \beta\ } & \mathrm{Inn}(K) \ \subseteq \ \mathrm{Aut}(K) \\
\downarrow & & \downarrow \\
1 & & 1
\end{array} \tag{22}$$

Lemma 29. *There is a well-defined* ***untwisted*** *left action of $\widetilde{G}$ on M by $\mathbb{Z}[K]$-module automorphisms*

$$\gamma\colon \widetilde{G} \longrightarrow \mathrm{Aut}_{\mathbb{Z}[K]}(M)$$

given by the formula $\gamma(\widetilde{g})(m) = \alpha(\pi(\widetilde{g}))(m).\theta(\widetilde{g})$.

Proof. It is clear that, for fixed $\widetilde{g} \in \widetilde{G}$, the formula $\alpha(\pi(\widetilde{g}))(-).\theta(\widetilde{g})$ defines a $\mathbb{Z}$-module automorphism of M. We therefore just have to check two things:

- The automorphism $\alpha(\pi(\widetilde{g}))(-).\theta(\widetilde{g})$ of M commutes with the right action of K.
- The function $\widetilde{g} \mapsto \alpha(\pi(\widetilde{g}))(-).\theta(\widetilde{g})$ is a group homomorphism.

For the first point, let $\widetilde{g} \in \widetilde{G}$, $m \in M$ and $h \in K$. We have

$$\begin{aligned}
\alpha(\pi(\widetilde{g}))(m.h).\theta(\widetilde{g}) &= \alpha(\pi(\widetilde{g}))(m).\beta(\pi(\widetilde{g}))(h).\theta(\widetilde{g}) \\
&= \alpha(\pi(\widetilde{g}))(m).\theta(\widetilde{g}).h.\theta(\widetilde{g})^{-1}.\theta(\widetilde{g}) \\
&= \alpha(\pi(\widetilde{g}))(m).\theta(\widetilde{g}).h,
\end{aligned}$$

where the first equality holds by our compatibility assumption between the actions α and β, and the second one holds by commutativity of (22). This says precisely that $\alpha(\pi(\widetilde{g}))(-).\theta(\widetilde{g})$ commutes with the right action of K.

For the second point, let $\widetilde{g}_1, \widetilde{g}_2 \in \widetilde{G}$ and $m \in M$. We have

$$\begin{aligned}
\alpha(\pi(\widetilde{g}_2))\Big(\alpha(\pi(\widetilde{g}_1))(m).\theta(\widetilde{g}_1)\Big).\theta(\widetilde{g}_2) &= \alpha(\pi(\widetilde{g}_2))\Big(\alpha(\pi(\widetilde{g}_1))(m)\Big).\beta(\pi(\widetilde{g}_2))(\theta(\widetilde{g}_1)).\theta(\widetilde{g}_2) \\
&= \alpha(\pi(\widetilde{g}_2))\Big(\alpha(\pi(\widetilde{g}_1))(m)\Big).\theta(\widetilde{g}_2).\theta(\widetilde{g}_1).\theta(\widetilde{g}_2)^{-1}.\theta(\widetilde{g}_2) \\
&= \alpha(\pi(\widetilde{g}_2\widetilde{g}_1))(m).\theta(\widetilde{g}_2\widetilde{g}_1),
\end{aligned}$$

where, again, the first equality holds by our compatibility assumption between the actions α and β, the second one holds by commutativity of (22) and the third one holds since α, π and θ are homomorphisms. This says precisely that $\widetilde{g} \mapsto \alpha(\pi(\widetilde{g}))(-).\theta(\widetilde{g})$ is a group homomorphism. ■

4.2 Untwisting in bundles.

We will need a natural *fibrewise* version of Lemma 29, whose proof is identical (one just has to think of bundles of modules instead of modules). Let X be a space and let $\xi\colon \mathcal{M} \to X$ be a bundle of $\mathbb{Z}[K]$-modules. Suppose that G has a twisted left action on $\mathcal{M}$. Precisely, this means a pair of homomorphisms

$$\alpha\colon G \longrightarrow \mathrm{Homeo}(\mathcal{M})$$
$$\beta\colon G \longrightarrow \mathrm{Aut}(K)$$

such that, for each $g \in G$, $x \in X$, $m \in \xi^{-1}(x)$ and $h \in K$, we have:

- $\alpha(g)$ preserves the fibres of ξ,
- the restriction of $\alpha(g)$ to each fibre is a $\mathbb{Z}$-linear automorphism,
- $\alpha(g)(m.h) = \alpha(g)(m).\beta(g)(h)$.

(By definition, such an action is an *untwisted* left action by automorphisms of bundles of $\mathbb{Z}[K]$-modules exactly when β is the trivial action.)

As before, assume that the action β of G on K is by *inner* automorphisms and define the central extension $\widetilde{G}$ of G as in (22).

Lemma 30. *There is a well-defined* ***untwisted*** *left action of $\widetilde{G}$ on $\xi\colon \mathcal{M} \to X$ by automorphisms of bundles of $\mathbb{Z}[K]$-modules*

$$\gamma\colon \widetilde{G} \longrightarrow \mathrm{Aut}_{\mathbb{Z}[K]}(\xi\colon \mathcal{M} \to X)$$

given by the formula $\gamma(\widetilde{g})(m) = \alpha(\pi(\widetilde{g}))(m).\theta(\widetilde{g})$.

4.3 Rescaling.

Once we have obtained an untwisted representation of $\widetilde{G}$, the following "rescaling" lemma gives (under conditions) a trick to ensure that it descends to an (untwisted) representation of G. It is abstracted from §2 of [12].

Suppose that we have a central extension

$$1 \to \mathbb{Z} \xrightarrow{\iota} \widetilde{G} \xrightarrow{\pi} G \to 1$$

together with a representation ρ of $\widetilde{G}$ on an R-module V. Suppose also that there exists a quotient $q\colon \widetilde{G} \twoheadrightarrow \mathbb{Z}$ such that $q(\iota(1)) = k \neq 0$ and that $\rho(\iota(1)) = \mathrm{id}_V.\lambda$, for an element $\lambda \in R^*$ that admits a k-th root, in other words there exists $\mu \in R^*$ with $\mu^k = \lambda$. Then the following lemma is immediate.

Lemma 31. *Under the above conditions, the representation of $\widetilde{G}$ on the R-module V given by the formula*

$$\widetilde{g} \longmapsto \rho(\widetilde{g}).\mu^{-q(\widetilde{g})}$$

descends to a representation of G.

5 Constructing the representations

We now put everything together to construct the first three homological representations of the Main Theorem: the twisted representation (A) of $\mathfrak{M}(\Sigma)$ is constructed in §5.2, its (untwisted) restriction (B) to the Chillingworth subgroup is described in §5.3 and the (untwisted) representation (C) of a $\mathbb{Z}$-central extension of the Torelli group is constructed in §5.4. The representation (D) of a double covering of $\mathfrak{M}(\Sigma)$ using the Schrödinger representation will be constructed in §6. Before all of this, we discuss some generalities about twisted representations in §5.1.

5.1 Twisted representations of groups.

A representation of a group G over a ring R is a functor $G \to \mathrm{Mod}_R$, where we are considering G as a single-object groupoid. A twisted representation of G is similar, except that the group G has been "spread out" over many objects according to a given action of G on a set X.

Definition 32. Let G be a group equipped with a left action $a\colon G \to \mathrm{Sym}(X)$. The *action groupoid* $\mathrm{Ac}(G \curvearrowright X)$ is the groupoid whose objects are $\mathrm{im}(a)$, in other words those symmetries of X that are induced by some element of G, and whose morphisms $\sigma \to \tau$ are the elements $a^{-1}(\tau\sigma^{-1}) \subseteq G$. Composition is given by multiplication in the group.

Remark 33. If instead G has a *right* action on X, then $a\colon G \to \mathrm{Sym}(X)$ will be an anti-homomorphism, rather than a homomorphism, and the morphisms $\sigma \to \tau$ of the analogous action groupoid $\mathrm{Ac}(X \curvearrowleft G)$ are the elements $a^{-1}(\sigma\tau^{-1}) \subseteq G$.

Remark 34. For any object σ of $\mathrm{Ac}(G \curvearrowright X)$, the automorphism group of σ is equal to $\ker(a) \subseteq G$. Moreover, the set of all morphisms with target τ is naturally identified with the whole group G.

Example 35. If the G-action on X is trivial, then $\mathrm{Ac}(G \curvearrowright X)$ is just G considered as a one-object groupoid. In the case where $X = G$ acting on itself by left-multiplication, $\mathrm{Ac}(G \curvearrowright X)$ is sometimes known as the *translation groupoid* of G.

Definition 36. A *twisted representation* of a group G over a ring R is a left G-set X and a functor $\mathrm{Ac}(G \curvearrowright X) \to \mathrm{Mod}_R$, where Mod_R is the category of R-modules. In particular, an ordinary (untwisted) representation is the special case where the action of G on X is trivial. Similarly for any other flavour of representations given by a category $\mathcal{C}$, such as the category of Hilbert spaces (unitary representations), or the category of bundles of R-modules (fibrewise representations over R), etc.: a *twisted representation* of G of this flavour is a left G-set X and a functor $\mathrm{Ac}(G \curvearrowright X) \to \mathcal{C}$.

Remark 37. In the previous section, we considered a notion of *twisted action* of a group G. This may be thought of as a special case of a twisted representation in the sense of Definition 36, as we now explain. Recall that a twisted left action of G on a right

$\mathbb{Z}[K]$-module M is a $\mathbb{Z}$-linear left action α of G on M together with a left action β of G on K, such that $\alpha(g)(m.h) = \alpha(g)(m).\beta(g)(h)$, in other words, β measures the failure of α to respect the right action of K on M.

In this situation, we obtain a twisted representation of G over the ring $\mathbb{Z}[K]$ by taking $X = K$ with G-action given by β, and we define the functor

$$\mathrm{Ac}(G \curvearrowright K) \longrightarrow \mathrm{Mod}_{\mathbb{Z}[K]}$$

sending the object $\sigma \in \mathrm{im}(\beta) \subseteq \mathrm{Aut}(K)$ to the $\mathbb{Z}[K]$-module M_σ defined as follows: as $\mathbb{Z}$-modules, $M_\sigma = M$, but the right K-action on M_σ is given by $m \cdot_\sigma h = m.\sigma(h)$, where $.$ denotes the right K-action on M. Morphisms of $\mathrm{Ac}(G \curvearrowright K)$ are elements of G, and the functor is defined on them by α. The compatibility condition between α and β assumed above is exactly what is needed to imply that this gives a well-defined functor into the category of $\mathbb{Z}[K]$-modules (not just $\mathbb{Z}$-modules).

Remark 38. To formulate the notion of *twisted representations* of a group, one may either break apart the domain group into a groupoid (as above) or one may enlarge the target category from Mod_R to a larger category that also contains *twisted* R-linear homomorphisms (this is essentially the viewpoint taken in §4). These two viewpoints are related, as explained in Remark 37. The former viewpoint is most convenient for us in this section, whereas the latter viewpoint is more convenient if one wishes to construct twisted representations of categories (rather than just groups).

For example, Soulié and the second author study in [33] a general construction of twisted representations of families of groups, where *families of groups* are encoded in appropriate Quillen bracket categories, and twisted representations are viewed as functors into $\mathrm{Mod}_\bullet$, the category of all modules over all rings (see [33, §2] for precise details). In particular, the family of mapping class groups $\{\mathfrak{M}(\Sigma_{g,1})\}_{g\geq 1}$ may be encoded as the automorphism groups of a category $\mathfrak{U}\mathcal{M}_2^+$. In the case when $n \geq 3$ (so that the quotient $\phi\colon \pi_1(\mathcal{C}_n(\Sigma)) \twoheadrightarrow \mathcal{H}$ is the universal 2-nilpotent quotient), certain twisted representations $\mathfrak{U}\mathcal{M}_2^+ \to \mathrm{Mod}_\bullet$ constructed in [33, §5.4.1], when restricted to a particular automorphism group $\mathfrak{M}(\Sigma) = \mathfrak{M}(\Sigma_{g,1})$, correspond to the twisted representations (30) that we shall construct below.

5.2 A twisted representation of the mapping class group.

Recall from §3 (Propositions 13, 24 and 25) that we have a representation

$$\Psi\colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}(\mathcal{H})$$

such that $\ker(\Psi) = \mathrm{Chill}(\Sigma)$ and $\Psi^{-1}(\mathrm{Inn}(\mathcal{H})) = \mathfrak{T}(\Sigma)$ for $g \geq 1$.

The quotient homomorphism $\phi\colon \mathbb{B}_n(\Sigma) \twoheadrightarrow \mathcal{H}$ (§1) corresponds to a regular covering $\widetilde{\mathcal{C}}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$. Let $f \in \mathfrak{M}(\Sigma)$, $f_{\mathcal{H}}$ be its action on the Heisenberg group and $\mathcal{C}_n(f)$ be the action (up to isotopy) on the configuration space $\mathcal{C}_n(\Sigma)$. From Proposition 13 we know that $\mathcal{C}_n(\Sigma)_\sharp = f_{\mathbb{B}_n(\Sigma)}$ preserves $\ker(\phi)$ which implies that there exists a unique lift of $\mathcal{C}_n(f)$ fixing the base point

$$\widetilde{\mathcal{C}}_n(f)\colon \widetilde{\mathcal{C}}_n(\Sigma) \longrightarrow \widetilde{\mathcal{C}}_n(\Sigma)\ . \tag{23}$$

The action of $\widetilde{\mathcal{C}}_n(f)$ on the fibre over the base point identified with $\mathcal{H}$ coincides with $f_{\mathcal{H}}$, and for the deck action of $h \in \mathcal{H}$ on $x \in \widetilde{\mathcal{C}}_n(\Sigma)$ we have the twisting formula

$$\widetilde{\mathcal{C}}_n(f)(x \cdot h) = \widetilde{\mathcal{C}}_n(f) \cdot f_{\mathcal{H}}(h) \ .$$

The induced action on the singular complex $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))$ is twisted $\mathbb{Z}[\mathcal{H}]$-linear, which can be formulated as a $\mathbb{Z}[\mathcal{H}]$-linear isomorphism (up to chain homotopy)

$$\mathcal{S}_*(\widetilde{\mathcal{C}}_n(f))\colon \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{f_{\mathcal{H}}^{-1}} \longrightarrow \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \ .$$

Here the subscript on the domain means that the right action of $\mathcal{H}$ is twisted with $f_{\mathcal{H}}^{-1}$, just as in Remark 37. The result for $\mathbb{Z}[\mathcal{H}]$-local homology is a $\mathbb{Z}[\mathcal{H}]$-linear isomorphism

$$\mathcal{C}_n(f)_*\colon H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_{f_{\mathcal{H}}^{-1}} \longrightarrow H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H}) \ . \qquad (24)$$

More generally, if V is a left representation of the Heisenberg group over a ring R, then we obtain an R-linear isomorphism

$$\mathcal{C}_n(f)_*\colon H_*^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{f_{\mathcal{H}}}V\big) \longrightarrow H_*^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\big) \ , \qquad (25)$$

where the left-hand homology group is obtained from the chain complex

$$\mathcal{S}_*\big(\widetilde{\mathcal{C}}_n(\Sigma)\big)_{f_{\mathcal{H}}^{-1}} \underset{\mathbb{Z}[\mathcal{H}]}{\otimes} V \ \cong \ \mathcal{S}_*\big(\widetilde{\mathcal{C}}_n(\Sigma)\big) \underset{\mathbb{Z}[\mathcal{H}]}{\otimes} {}_{f_{\mathcal{H}}}V \ .$$

(Here, "obtained from" means in more detail that we consider the quotients of this chain complex given by the relative singular complexes for all subspaces of $\widetilde{\mathcal{C}}_n(\Sigma)$ of the form $\pi^{-1}(\mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \cup (\mathcal{C}_n(\Sigma) \smallsetminus T))$ for compact subsets $T \subset \mathcal{C}_n(\Sigma)$, where π denotes the covering map $\widetilde{\mathcal{C}}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$, take the homology of each of these quotients and then take the inverse limit of this diagram.)

An equivalent way to see that we obtain (24) and (25) is as follows. For a quotient homomorphism $q\colon \pi_1(\mathcal{C}_n(\Sigma)) \twoheadrightarrow Q$ let us write $\widetilde{\mathcal{C}}_n(\Sigma)^q$ for the corresponding regular Q-covering of $\mathcal{C}_n(\Sigma)$, considered as a space with a right Q-action. In this notation, the lifted action (23) is of the form

$$\widetilde{\mathcal{C}}_n(\Sigma)^{f_{\mathcal{H}} \circ \phi} \longrightarrow \widetilde{\mathcal{C}}_n(\Sigma)^{\phi} \qquad (26)$$

and commutes with the right $\mathcal{H}$-action on the source and target. Note that the right $\mathcal{H}$-action on the left-hand space is twisted by $f_{\mathcal{H}}^{-1}$ compared with its right action on the right-hand space. This is because the action of

$$\pi_1(\mathcal{C}_n(\Sigma)) \xrightarrow{\ \phi\ } \mathcal{H} \xrightarrow{\ f_{\mathcal{H}}\ } \mathcal{H} \ni h$$

is given by sending h backwards along $f_{\mathcal{H}}$ and then applying the untwisted action. Thus, applying relative Borel-Moore homology to (26), we obtain (24) with $\mathbb{Z}[\mathcal{H}]$-local coefficients and (25) with V-local coefficients.

Slightly more generally, for $\tau \in \mathrm{Aut}(\mathcal{H})$, the action $\mathcal{C}_n(f)\colon \mathcal{C}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$ lifts to

$$\widetilde{\mathcal{C}}_n(\Sigma)^{\tau^{-1}\circ f_{\mathcal{H}}\circ\phi} \longrightarrow \widetilde{\mathcal{C}}_n(\Sigma)^{\tau^{-1}\circ\phi} \tag{27}$$

and, applying relative Borel-Moore homology, we obtain

$$H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_{f_{\mathcal{H}}^{-1}\circ\tau} \longrightarrow H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_{\tau} \tag{28}$$

with $\mathbb{Z}[\mathcal{H}]$-local coefficients and

$$H_*^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{\tau^{-1}\circ f_{\mathcal{H}}}V\big) \longrightarrow H_*^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{\tau^{-1}}V\big) \tag{29}$$

with V-local coefficients.

Summarising this discussion, we have shown:

Theorem 39 (Part (A) of the Main Theorem.)**.** *Associated to any representation V of $\mathcal{H}$ over R, there is a well-defined twisted representation*

$$\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}) \longrightarrow \mathrm{Mod}_R \tag{30}$$

in the sense of Definition 36.

Proof. The object $\tau\colon \mathcal{H} \to \mathcal{H}$ of $\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H})$ is sent to the R-module

$$H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); {}_{\tau^{-1}}V\big)$$

and the morphism $f\colon f_{\mathcal{H}}^{-1} \circ \tau \to \tau$ is sent to the R-linear isomorphism (29). ∎

Remark 40. The functor (30) factors through the category of pairs of spaces equipped with local systems over $\mathbb{Z}[\mathcal{H}]$, which we denote by $\mathrm{Top}^2_{\mathbb{Z}[\mathcal{H}]}$. To see this, we send the object τ of $\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H})$ to the bundle of $\mathbb{Z}[\mathcal{H}]$-modules obtained by applying the free abelian group functor fibrewise to $\widetilde{\mathcal{C}}_n(\Sigma)^{\tau^{-1}\circ\phi}$, together with the subspace $\mathcal{C}_n(\Sigma, \partial^-(\Sigma)) \subset \mathcal{C}_n(\Sigma)$. (We recall that, under mild conditions that are satisfied here, local systems over $\mathbb{Z}[\mathcal{H}]$ may be thought of as bundles of $\mathbb{Z}[\mathcal{H}]$-modules.) We send the morphism $f\colon f_{\mathcal{H}}^{-1} \circ \tau \to \tau$ to the result of applying the free abelian group functor fibrewise to (27). This defines a functor

$$\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}) \longrightarrow \mathrm{Top}^2_{\mathbb{Z}[\mathcal{H}]} \tag{31}$$

and the remainder of the construction then consists in composing (31) with the fibrewise tensor product functor $- \otimes_{\mathbb{Z}[\mathcal{H}]} V\colon \mathrm{Top}^2_{\mathbb{Z}[\mathcal{H}]} \to \mathrm{Top}^2_R$ and relative Borel-Moore homology functor $H_n^{BM}\colon \mathrm{Top}^2_R \to \mathrm{Mod}_R$.

5.3 Restricting to the Chillingworth subgroup.

As mentioned in Definition 32, the automorphism groups of the groupoid $\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H})$ are all isomorphic to the kernel of the action $\Psi\colon \mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}$, which is the Chillingworth subgroup $\mathrm{Chill}(\Sigma)$, by Proposition 24. Restricting (30) to the automorphism group of the object $\mathrm{id}\colon \mathcal{H} \to \mathcal{H}$ of $\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H})$ therefore gives us an untwisted representation

$$\mathrm{Chill}(\Sigma) \longrightarrow \mathrm{Mod}_R$$

of the Chillingworth group. Concretely, the underlying R-module of this representation is the relative V-local Borel-Moore homology module

$$H_*^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\bigr),$$

and each $f \in \mathrm{Chill}(\Sigma)$ is sent to (29) with $\tau = f_{\mathcal{H}} = \mathrm{id}$. Thus we have shown:

Theorem 41 (Part (B) of the Main Theorem.)**.** *Associated to any representation V of $\mathcal{H}$ over R, there is a well-defined representation*

$$\mathrm{Chill}(\Sigma) \longrightarrow \mathrm{Aut}_R\bigl(H_n^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\bigr)\bigr), \tag{32}$$

which is the restriction of (30)*.*

Remark 42. Recall from §2 (Theorem 10) that the R-module $H_n^{BM}\bigl(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\bigr)$ is naturally isomorphic to a direct sum of $\binom{2g+n-1}{n}$ copies of V. In particular, if V is a free R-module of rank N, the right-hand side of (32) may be written as $GL_{N\binom{2g+n-1}{n}}(R)$.

5.4 The Torelli group.

We now restrict to the Torelli group $\mathfrak{T}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$. In this case we have $\Psi^{-1}(\mathrm{Inn}(\mathcal{H})) = \mathfrak{T}(\Sigma)$ by Proposition 25. We may therefore pull back the $\mathbb{Z}$-central extension

$$1 \to \mathbb{Z} \cong \mathcal{Z}(\mathcal{H}) \longrightarrow \mathcal{H} \longrightarrow \mathrm{Inn}(\mathcal{H}) \to 1$$

along the homomorphism $\Psi\colon \mathfrak{T}(\Sigma) \to \mathrm{Inn}(\mathcal{H})$ to obtain a $\mathbb{Z}$-central extension

$$1 \to \mathbb{Z} \longrightarrow \widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathfrak{T}(\Sigma) \to 1$$

and a homomorphism

$$\widetilde{\Psi}\colon \widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathcal{H}$$

lifting Ψ. We will this to "untwist" the representation (30) on the Torelli group.

For an element $h \in \mathcal{H}$, denote by $c_h = h{-}h^{-1}$ the corresponding inner automorphism $c_h \in \mathrm{Inn}(\mathcal{H})$. One may check that the isomorphism

$$- \cdot h\colon \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{c_h} \longrightarrow \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \tag{33}$$

of singular chain complexes given by the right-action of h is $\mathbb{Z}[\mathcal{H}]$-linear. The construction of §5.2, shifted by $f_{\mathcal{H}}$, sends each $f \in \mathfrak{T}(\Sigma)$ to a $\mathbb{Z}[\mathcal{H}]$-linear isomorphism

$$\mathcal{S}_*(\widetilde{\mathcal{C}}_n(f))\colon \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \longrightarrow \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{f_{\mathcal{H}}}\ , \tag{34}$$

where $f_{\mathcal{H}} = \Psi(f)$. For $\widetilde{f} \in \widetilde{\mathfrak{T}}(\Sigma)$ we therefore obtain a $\mathbb{Z}[\mathcal{H}]$-linear automorphism

$$\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \xrightarrow{(34)} \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))_{f_{\mathcal{H}}} \xrightarrow{(33)} \mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \ , \tag{35}$$

where we take $h = \widetilde{\Psi}(\widetilde{f})$ in (33). This defines an untwisted action of the central extension $\widetilde{\mathfrak{T}}(\Sigma)$ of the Torelli group on the singular chain complex $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma))$, and thus also on $\mathcal{S}_*(\widetilde{\mathcal{C}}_n(\Sigma)) \otimes_{\mathbb{Z}[\mathcal{H}]} V$ for any left $\mathcal{H}$-representation V. Moreover, we may repeat this construction for the *relative* singular chain complex with respect to any subspace of $\mathcal{C}_n(\Sigma)$, and this is compatible with taking inverse limits, and so we obtain an untwisted action of the central extension $\widetilde{\mathfrak{T}}(\Sigma)$ of the Torelli group on the relative V-local Borel-Moore homology

$$H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\big)$$

for any left $\mathcal{H}$-representation V. Thus we have shown:

Theorem 43 (Part (C) of the Main Theorem.)**.** *Associated to any representation V of $\mathcal{H}$ over R, there is a well-defined representation*

$$\widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathrm{Aut}_R\big(H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\big)\big), \tag{36}$$

where $\widetilde{\mathfrak{T}}(\Sigma)$ is a central extension by $\mathbb{Z}$ of the Torelli group $\mathfrak{T}(\Sigma)$.

Remark 42 applies also in this setting; in particular, $H_n^{BM}\big(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V\big)$ is a free R-module whenever V is a free R-module.

Remark 44. We have described the untwisting at the level of the singular chain complex, but it may in fact be done already at the level of spaces equipped with local systems over $\mathbb{Z}[\mathcal{H}]$, as explain in §4; see especially Lemma 30. (The verification that the composition (35) really gives a well-defined action of $\widetilde{\mathfrak{T}}(\Sigma)$ is essentially equivalent to the proof of Lemma 29.) In particular, the central extension $\widetilde{\mathfrak{T}}(\Sigma)$ of the Torelli group is the pullback in diagram (22) in the special case where $K = \mathcal{H}$ and $G = \mathfrak{T}(\Sigma)$.

Remark 45. In [23], Johnson defined a homomorphism $t\colon \mathfrak{T}(\Sigma) \to H = H_1(\Sigma)$, which was implicit in [15] and called it the Chillingworth homomorphism. Its value on a genus k bounding pair generator $\tau_\gamma\tau_\delta^{-1}$ is equal to $2kc$ where c is the class of γ with orientation given by the surface representing the homology between γ and δ. Its image is $2H$ and so we obtain a 2-cocycle Ω on $\mathfrak{T}(\Sigma)$ with values in $\mathbb{Z}$ by the formula $\Omega(f,g) = \frac{1}{4}\omega(t_f, t_g)$, where ω is the symplectic 2-cocycle on H. Under the identifications $H \cong \mathcal{H}/\mathcal{Z}(\mathcal{H}) \cong \mathrm{Inn}(\mathcal{H})$, this is precisely the action $\Psi\colon \mathfrak{M}(\Sigma) \to \mathrm{Aut}(\mathcal{H})$ restricted to the Torelli group (which acts on $\mathcal{H}$ by inner automorphisms by Proposition 25). Thus the central extension $\widetilde{\mathfrak{T}}(\Sigma)$, which was defined abstractly by pulling back the central extension

$$1 \to \mathbb{Z} \cong \mathcal{Z}(\mathcal{H}) \hookrightarrow \mathcal{H} \twoheadrightarrow \mathcal{H}/\mathcal{Z}(\mathcal{H}) \cong \mathrm{Inn}(\mathcal{H}) \to 1$$

along the inner action $\Psi|_{\mathfrak{T}(\Sigma)}\colon \mathfrak{T}(\Sigma) \to \mathrm{Inn}(\mathcal{H})$, see diagram (22), may be described more explicitly as the central extension of the Torelli group associated to Ω. In other words, we have $\widetilde{\mathfrak{T}}(\Sigma) = \mathbb{Z} \times \mathfrak{T}(\Sigma)$ as a set, and $(k, f)(l, g) = (k + l + \Omega(f, g), fg)$. Moreover, we have a lift of the Chillingworth homomorphism

$$\widetilde{t}\colon \widetilde{\mathfrak{T}}(\Sigma) \longrightarrow \mathcal{H}$$

given by the pullback construction (22), which may be described by the formula $\widetilde{t}(k, f) = (4k, t_f)$. This is the homomorphism denoted by $\widetilde{\Psi}$ above.

6 Untwisting on the full mapping class group via Schrödinger

The Heisenberg group $\mathcal{H}$ can be realised as a group of matrices, which gives a faithful finite dimensional representation, defined as follows:

$$\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right) \mapsto \begin{pmatrix} 1 & p & \frac{k+p\cdot q}{2} \\ 0 & I_g & q \\ 0 & 0 & 1 \end{pmatrix},$$

where $p = (p_i)$ is a row vector and $q = (q_i)$ is a column vector. This matrix form is often given as the definition of the Heisenberg group so that we may call this representation the tautological one. This is the representation (1) from the introduction.

Another well-known representation, which is infinite dimensional and unitary, is the *Schrödinger representation*, which is parametrised by the Planck constant, a non-zero real number $\hbar$. The right action on the Hilbert space $L^2(\mathbb{R}^g)$ is given by the following formula:

$$\left[\Pi_\hbar \left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right) \psi\right](s) = e^{i\hbar\frac{k+p\cdot q}{2}} e^{iq\cdot s} \psi(s + \hbar p). \tag{37}$$

This is the representation (1) from the introduction.

The Schrödinger representation occupies a special place in the representation theory of the Heisenberg group, and in this section we explain how to leverage its properties to construct an untwisted representation on the full mapping class group $\mathfrak{M}(\Sigma)$, after passing to a central extension. For comparison, recall that, in the previous section, we constructed an untwisted representation of (a central extension of) the Torelli group $\mathfrak{T}(\Sigma) \subseteq \mathfrak{M}(\Sigma)$ associated to *any* representation V of the Heisenberg group. The difference in this section is that we focus only on the special case where V is the *Schrödinger representation*, but as a consequence we are able to untwist the representation on the full mapping class group.

The *continuous Heisenberg group* is defined very similarly to the discrete Heisenberg group. As a set it is $\mathbb{R} \times H_1(\Sigma; \mathbb{R})$, where $\Sigma = \Sigma_{g,1}$ as before, with multiplication given by $(s, x).(t, y) = (s + t + \omega(x, y), x + y)$, where ω is the intersection form on $H_1(\Sigma; \mathbb{R})$. We denote it by $\mathcal{H}_\mathbb{R}$ and note that the discrete Heisenberg group $\mathcal{H}$ is naturally a subgroup of $\mathcal{H}_\mathbb{R}$. As explained in Appendix B, there is a natural inclusion

$$\mathrm{Aut}^+(\mathcal{H}) \longleftrightarrow \mathrm{Aut}^+(\mathcal{H}_\mathbb{R}),$$

denoted by $\varphi \mapsto \varphi_{\mathbb{R}}$, such that $\varphi_{\mathbb{R}}$ is an extension of φ (see (61)).

As an alternative to the explicit formula (37), the Schrödinger representation may also be defined more abstractly as follows. First note that $\mathcal{H}_{\mathbb{R}}$ may be written as a semi-direct product

$$\mathcal{H}_{\mathbb{R}} = \mathbb{R}\{(1,0),(0,a_1),\dots,(0,a_g)\} \rtimes \mathbb{R}\{(0,b_1),\dots,(0,b_g)\},$$

where $a_1,\dots,a_g,b_1,\dots,b_g$ form a symplectic basis for $H_1(\Sigma;\mathbb{R})$. Fix a real number $\hbar > 0$. There is a one-dimensional complex unitary representation

$$\mathbb{R}\{(1,0),(0,a_1),\dots,(0,a_g)\} \longrightarrow \mathbb{S}^1 = U(1)$$

defined by $(t,x) \mapsto e^{\hbar it/2}$. This may then be induced to a complex unitary representation of the whole group $\mathcal{H}_{\mathbb{R}}$ on the complex Hilbert space $L^2(\mathbb{R}\{(0,b_1),\dots,(0,b_g)\}) = L^2(\mathbb{R}^g)$. This is the Schrödinger representation of $\mathcal{H}_{\mathbb{R}}$. From now on, let us denote this representation by

$$W\colon \mathcal{H}_{\mathbb{R}} \longrightarrow U(L^2(\mathbb{R}^g)). \tag{38}$$

We will usually not make the dependence on $\hbar$ explicit in the notation; in particular we write W instead of $W_\hbar$. The main properties of W that we shall need are the following.

Theorem 46 (The *Stone–von Neumann theorem*; [27, page 19])**.**
(a) *The representation (38) is irreducible.*
(b) *If H' is a complex Hilbert space and*

$$W'\colon \mathcal{H}_{\mathbb{R}} \longrightarrow U(H')$$

is a unitary representation such that $W'(t,0) = e^{\hbar it/2}.\mathrm{id}_{H'}$ for all $t \in \mathbb{R}$, then there is another Hilbert space H'' and an isomorphism $\kappa\colon H' \to L^2(\mathbb{R}^g) \otimes H''$ such that, for any $(t,x) \in \mathcal{H}_{\mathbb{R}}$, the following diagram commutes:

$$\begin{array}{ccc} H' & \xrightarrow{\quad\kappa\quad} & L^2(\mathbb{R}^g)\otimes H'' \\ \Big\downarrow{\scriptstyle W'(t,x)} & & \Big\downarrow{\scriptstyle W(t,x)\otimes \mathrm{id}_{H''}} \\ H' & \xrightarrow{\quad\kappa\quad} & L^2(\mathbb{R}^g)\otimes H''. \end{array}$$

Corollary 47. *If $W'\colon \mathcal{H}_{\mathbb{R}} \to U(L^2(\mathbb{R}^g))$ is an irreducible unitary representation such that $W'(t,0) = e^{\hbar it/2}.\mathrm{id}$ for all $t \in \mathbb{R}$, then there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{H}_{\mathbb{R}} & \xrightarrow{\quad W\quad} & U(L^2(\mathbb{R}^g)) \\ & {\scriptstyle W'}\searrow & \Big\downarrow{\scriptstyle u.-.u^{-1}} \\ & & U(L^2(\mathbb{R}^g)) \end{array}$$

for some element $u \in U(L^2(\mathbb{R}^g))$, which is unique up to rescaling by an element of $\mathbb{S}^1$.

Proof. Apply Theorem 46 and note that $\dim(H'') = 1$ since W' is irreducible. The unitary isomorphism κ together with any choice of unitary isomorphism $L^2(\mathbb{R}^g) \otimes \mathbb{R} \cong L^2(\mathbb{R}^g)$ give an element u as claimed. To see uniqueness up to a scalar in $\mathbb{S}^1$, note that any two such elements u differ by an automorphism of the irreducible representation W, which must therefore be a scalar (in $\mathbb{C}^*$) multiple of the identity, by Schur's lemma. Moreover, since W is unitary, this scalar must lie in $\mathbb{S}^1 \subset \mathbb{C}^*$. ■

Definition 48. Denote by $PU(L^2(\mathbb{R}^g)) = U(L^2(\mathbb{R}^g))/\mathbb{S}^1$ the *projective unitary group* of the Hilbert space $L^2(\mathbb{R}^g)$. Since scalar multiples of the identity are central, this fits into a central extension

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow U(L^2(\mathbb{R}^g)) \longrightarrow PU(L^2(\mathbb{R}^g)) \longrightarrow 1. \tag{39}$$

We denote by $\omega_{PU} \colon PU(L^2(\mathbb{R}^g)) \times PU(L^2(\mathbb{R}^g)) \to \mathbb{S}^1$ a choice of 2-cocycle determining this central extension; in other words we may write $U(L^2(\mathbb{R}^g)) \cong \mathbb{S}^1 \times PU(L^2(\mathbb{R}^g))$ with multiplication given by $(s,g)(t,h) = (s.t.\omega_{PU}(g,h), gh)$.

Definition 49. For an automorphism $\varphi \in \mathrm{Aut}(\mathcal{H}_{\mathbb{R}})$, Corollary 47 applied to the representation $W' = W \circ \varphi$ tells us that there is a unique $u = R(\varphi) \in PU(L^2(\mathbb{R}^g))$ such that $W \circ \varphi = R(\varphi).W.R(\varphi)^{-1}$. The assignment $\varphi \mapsto R(\varphi)$ defines a group homomorphism $\mathrm{Aut}(\mathcal{H}_{\mathbb{R}}) \to PU(L^2(\mathbb{R}^g))$, and one may check that it factors through the projection $\mathrm{Aut}(\mathcal{H}_{\mathbb{R}}) \twoheadrightarrow Sp_{2g}(\mathbb{R})$ that sends an automorphism of $\mathcal{H}_{\mathbb{R}}$ to the induced automorphism of its abelianisation $(\mathcal{H}_{\mathbb{R}})^{ab} = \mathbb{R}^{2g}$. We therefore obtain a projective representation

$$R \colon Sp_{2g}(\mathbb{R}) \longrightarrow PU(L^2(\mathbb{R}^g)). \tag{40}$$

This is the *Shale-Weil projective representation* of the symplectic group. (This is sometimes also called the *Segal-Shale-Weil projective representation*, see for example [27, page 53].) Pulling back the central extension (39) along the homomorphism (40), we obtain a central extension

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow \overline{Sp}_{2g}(\mathbb{R}) \longrightarrow Sp_{2g}(\mathbb{R}) \longrightarrow 1 \tag{41}$$

and a lifted representation

$$\overline{R} \colon \overline{Sp}_{2g}(\mathbb{R}) \longrightarrow U(L^2(\mathbb{R}^g)). \tag{42}$$

The group $\overline{Sp}_{2g}(\mathbb{R})$ is sometimes known as the *Mackey obstruction group* of the projective representation (40). Since (41) is pulled back from (39) along R, we may write $\overline{Sp}_{2g}(\mathbb{R}) \cong \mathbb{S}^1 \times Sp_{2g}(\mathbb{R})$ with multiplication given by $(s,g)(t,h) = (s.t.\omega_{Sp}(g,h), gh)$, where

$$\omega_{Sp} = \omega_{PU} \circ (R \times R) \colon Sp_{2g}(\mathbb{R}) \times Sp_{2g}(\mathbb{R}) \longrightarrow PU(L^2(\mathbb{R}^g)) \times PU(L^2(\mathbb{R}^g)) \longrightarrow \mathbb{S}^1.$$

Definition 50. The fundamental group of $Sp_{2g}(\mathbb{R})$ is infinite cyclic. It therefore has a unique connected double covering group, called the *metaplectic group*, which we denote by $Mp_{2g}(\mathbb{R})$.

For an explicit construction of $Mp_{2g}(\mathbb{R})$ as an extension of $Sp_{2g}(\mathbb{R})$, see [36, §2].

Proposition 51. *There is an embedding of topological groups $Mp_{2g}(\mathbb{R}) \hookrightarrow \overline{Sp}_{2g}(\mathbb{R})$.*

Proof. This follows from §1.7 of [27]. More precisely, it is proven in §1.7 of [27] that the cocycle ω_{Sp} takes values in the cyclic subgroup $\mathbb{Z}/8 \subseteq \mathbb{S}^1$ and that there is a function $s\colon Sp_{2g}(\mathbb{R}) \to \mathbb{Z}/4 \subseteq \mathbb{S}^1$ such that $\omega_{Sp}(g,h)^2 = s(g)^{-1}s(h)^{-1}s(gh)$ (formula 1.7.8 on page 70 of [27]). It then follows that the subset of $\overline{Sp}_{2g}(\mathbb{R})$ of those pairs (t,g) for which $t^2 = s(g)^{-1}$ is a (connected) subgroup. The projection onto $Sp_{2g}(\mathbb{R})$ restricted to this subgroup is a double covering, and so this subgroup must be the metaplectic group. ■

Definition 52. The *metaplectic mapping class group* $\widetilde{\mathfrak{M}}(\Sigma)$ is the double covering group of the mapping class group $\mathfrak{M}(\Sigma)$ pulled back from the double covering $Mp_{2g}(\mathbb{R}) \to Sp_{2g}(\mathbb{R})$ along the map

$$\mathfrak{M}(\Sigma) \twoheadrightarrow Sp_{2g}(\mathbb{Z}) \hookrightarrow Sp_{2g}(\mathbb{R}).$$

Putting this all together, we have a diagram

$$\begin{array}{ccccccccccc}
\widetilde{\mathfrak{M}}(\Sigma) & \xrightarrow{\widetilde{\Phi}} & \widetilde{\mathrm{Aut}}^{+}(\mathcal{H}) & \longrightarrow & Mp_{2g}(\mathbb{Z}) & \longrightarrow & Mp_{2g}(\mathbb{R}) & \longrightarrow & \overline{Sp}_{2g}(\mathbb{R}) & \xrightarrow{\overline{R}} & U(L^2(\mathbb{R}^g)) \\
\downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow & & \downarrow & \swarrow & & & \downarrow \\
\mathfrak{M}(\Sigma) & \xrightarrow{\Phi} & \mathrm{Aut}^{+}(\mathcal{H}) & \longrightarrow & Sp_{2g}(\mathbb{Z}) & \longrightarrow & Sp_{2g}(\mathbb{R}) & & \xrightarrow{\quad R \quad} & & PU(L^2(\mathbb{R}^g))
\end{array}$$

where all four squares are pullback squares, the left-hand four vertical maps are double coverings and the right-hand two vertical (or diagonal) maps are central extensions by the circle group.

Notation 53. We denote by

$$S\colon \widetilde{\mathfrak{M}}(\Sigma) \longrightarrow U(L^2(\mathbb{R}^g))$$

the composition along the top row of the diagram above. By abuse of notation, we write

$$W\colon \mathcal{H} \longrightarrow U(L^2(\mathbb{R}^g))$$

for the restriction of the Schrödinger representation (38) to the subgroup $\mathcal{H} \subset \mathcal{H}_{\mathbb{R}}$.

Using this notation, we record the following consequence of Definition 49:

Lemma 54. *For any element $\widetilde{g} \in \widetilde{\mathfrak{M}}(\Sigma)$ and any $h \in \mathcal{H}$, we have the following equation in $U(L^2(\mathbb{R}^g))$:*

$$S(\widetilde{g}).W(h).S(\widetilde{g})^{-1} = W(\Phi(\pi(\widetilde{g}))(h)). \tag{43}$$

We now use this to construct an *untwisted* unitary representation of the metaplectic mapping class group on Heisenberg homology of configuration spaces with coefficients in the Schrödinger representation.

Let $\widetilde{\mathcal{C}}_n(\Sigma) \to \mathcal{C}_n(\Sigma)$ denote the connected covering of $\mathcal{C}_n(\Sigma)$ corresponding to the kernel of the surjective homomorphism $\pi_1(\mathcal{C}_n(\Sigma)) \twoheadrightarrow \mathcal{H}$. This is a principal $\mathcal{H}$-bundle. Taking free abelian groups fibrewise, we obtain

$$\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)] \longrightarrow \mathcal{C}_n(\Sigma), \tag{44}$$

which is a bundle of right $\mathbb{Z}[\mathcal{H}]$-modules. Via the Schrödinger representation W, the Hilbert space $L^2(\mathbb{R}^g)$ becomes a left $\mathbb{Z}[\mathcal{H}]$-module, and we may take a fibrewise tensor product to obtain

$$\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)] \otimes_{\mathbb{Z}[\mathcal{H}]} L^2(\mathbb{R}^g) \longrightarrow \mathcal{C}_n(\Sigma), \tag{45}$$

which is a bundle of Hilbert spaces. There is a natural action of the mapping class group $\mathfrak{M}(\Sigma)$ (up to homotopy) on the base space $\mathcal{C}_n(\Sigma)$, and the induced action on $\pi_1(\mathcal{C}_n(\Sigma))$ preserves the kernel of the surjection $\pi_1(\mathcal{C}_n(\Sigma)) \twoheadrightarrow \mathcal{H}$ (Proposition 13), so that there is a well-defined twisted action of $\mathfrak{M}(\Sigma)$ on the bundle (44), in the following sense. There are homomorphisms

$$\alpha\colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}\big(\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]\big)$$
$$\Phi\colon \mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}(\mathcal{H})$$

such that, for any $g \in \mathfrak{M}(\Sigma)$, $h \in \mathcal{H}$ and $m \in \mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]$, we have

$$\alpha(g)(m.h) = \alpha(g)(m).\Phi(g)(h). \tag{46}$$

In other words, Φ measures the failure of α to be an action by $\mathbb{Z}[\mathcal{H}]$-module automorphisms. In the above formula, the target of α is the group of automorphisms of the bundle (44) (i.e. the group of self-homeomorphisms of the total space that preserve the fibres of the projection) that are $\mathbb{Z}$-linear (but not necessarily $\mathbb{Z}[\mathcal{H}]$-linear) on each fibre.

Theorem 55. *There is an action of the metaplectic mapping class group $\widetilde{\mathfrak{M}}(\Sigma)$ on (45) by Hilbert space bundle automorphisms*

$$\gamma\colon \widetilde{\mathfrak{M}}(\Sigma) \longrightarrow U\Big(\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)] \otimes_{\mathbb{Z}[\mathcal{H}]} L^2(\mathbb{R}^g) \longrightarrow \mathcal{C}_n(\Sigma)\Big)$$

given by the formula

$$\gamma(\widetilde{g})(m \otimes v) = \alpha(\pi(\widetilde{g}))(m) \otimes S(\widetilde{g})(v) \tag{47}$$

for all $\widetilde{g} \in \widetilde{\mathfrak{M}}(\Sigma)$, $m \in \mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]$ and $v \in L^2(\mathbb{R}^g)$.

Proof. This is similar in spirit to the proof of Lemma 30. The key property that needs to be verified is the following. Since we are taking the (fibrewise) tensor product over $\mathbb{Z}[\mathcal{H}]$, we have $m.h \otimes v = W(h)(v)$ for any $h \in \mathcal{H}$, $m \in \mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]$ and $v \in L^2(\mathbb{R}^g)$. (Note that we denote the right $\mathcal{H}$-action on the fibres of $\mathbb{Z}[\widetilde{\mathcal{C}}_n(\Sigma)]$ simply by juxtaposition, whereas the left $\mathcal{H}$-action on $L^2(\mathbb{R}^g)$ is the Schrödinger representation, denoted by W.)

We therefore have to verify that, for each fixed $\widetilde{g} \in \widetilde{\mathfrak{M}}(\Sigma)$, the formula (47) gives the same answer when applied to $m.h \otimes v$ or $m \otimes W(h)(v)$. To see this, we calculate:

$$\begin{aligned}
\gamma(\widetilde{g})(m.h \otimes v) &= \alpha(\pi(\widetilde{g}))(m.h) \otimes S(\widetilde{g})(v) && \text{by definition} \\
&= \alpha(\pi(\widetilde{g}))(m).\Phi(\pi(\widetilde{g}))(h) \otimes S(\widetilde{g})(v) && \text{by eq. (46)} \\
&= \alpha(\pi(\widetilde{g}))(m) \otimes W(\Phi(\pi(\widetilde{g}))(h))(S(\widetilde{g})(v)) && \text{since } \otimes \text{ is over } \mathbb{Z}[\mathcal{H}] \\
&= \alpha(\pi(\widetilde{g}))(m) \otimes S(\widetilde{g}) \circ W(h) \circ S(\widetilde{g})^{-1}(S(\widetilde{g})(v)) && \text{by eq. (43) [Lemma 54]} \\
&= \alpha(\pi(\widetilde{g}))(m) \otimes S(\widetilde{g})(W(h)(v)) && \text{simplifying} \\
&= \gamma(\widetilde{g})(m \otimes W(h)(v)). && \text{by definition}
\end{aligned}$$

This tells us that the formula (47) gives a well-defined bundle automorphism of (45) for each fixed $\widetilde{g} \in \widetilde{\mathfrak{M}}(\Sigma)$. It is then routine to verify that this bundle automorphism is $\mathbb{R}$-linear and unitary on fibres – i.e. it is an automorphism of bundles of Hilbert spaces – and that γ is a group homomorphism. ■

Theorem 56 (Part (D) of the Main Theorem.)**.** *The action of the mapping class group on the Borel-Moore homology of the configuration space $\mathcal{C}_n(\Sigma)$ with coefficients in the Schrödinger representation induces a well-defined complex unitary representation of the metaplectic mapping class group:*

$$\widetilde{\mathfrak{M}}(\Sigma) \longrightarrow U\big(H_n^{BM}(\mathcal{C}_n(\Sigma); L^2(\mathbb{R}^g))\big).$$

Proof. This is an immediate consequence of Theorem 55. In more detail, according to that theorem, we have a well-defined functor γ from the group $\widetilde{\mathfrak{M}}(\Sigma)$ to the category of spaces equipped with bundles of Hilbert spaces. On the other hand, twisted Borel-Moore homology $H_n^{BM}(-)$ is a functor from the category of spaces equipped with bundles of Hilbert spaces (and bundle maps whose underlying map of spaces is proper) to the category of Hilbert spaces. Composing these two functors, we obtain the desired unitary representation of $\widetilde{\mathfrak{M}}(\Sigma)$. ■

Remark 57. Recall that $\widetilde{\mathfrak{M}}(\Sigma)$ in Theorem 56 is the double covering of the mapping class group $\mathfrak{M}(\Sigma)$ pulled back from the metaplectic group covering the symplectic group. By [7, Lemma A.1, parts (i), (vi) and (xiv)], the homomorphism

$$H^2(Sp_{2g}(\mathbb{Z}); \mathbb{Z}/2) \longrightarrow H^2(\mathfrak{M}(\Sigma_g); \mathbb{Z}/2)$$

is an isomorphism when $g \geq 3$, and both sides are isomorphic to $\mathbb{Z}/2$ when $g \geq 4$ and to $(\mathbb{Z}/2)^2$ when $g = 3$. By homological stability for mapping class groups of surfaces (see for example [38, Theorem 1.2]), we may replace the closed surface Σ_g with the surface $\Sigma = \Sigma_{g,1}$, and the same statements hold for $g \geq 3$. Since the metaplectic group $Mp_{2g}(\mathbb{Z})$ is a non-trivial $\mathbb{Z}/2$-central extension of $Sp_{2g}(\mathbb{Z})$, it follows that:

- $\widetilde{\mathfrak{M}}(\Sigma)$ is a non-trivial $\mathbb{Z}/2$-central extension of $\mathfrak{M}(\Sigma)$ for $g \geq 3$;
- $\widetilde{\mathfrak{M}}(\Sigma)$ is the *unique* non-trivial $\mathbb{Z}/2$-central extension of $\mathfrak{M}(\Sigma)$ for $g \geq 4$.

Finite dimensional Schrödinger representation.

For $N \geq 2$, the finite dimensional Schrödinger representation is an action of the Heisenberg group $\mathcal{H}$ on $L^2((\mathbb{Z}/N)^g)$ which may be defined as follows:

$$\left[\varpi_N\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right)\psi\right](\varsigma) = e^{i\pi\frac{k+p\cdot q}{N}} e^{i\frac{2\pi}{N} q\cdot\varsigma}\psi(\varsigma + p). \tag{48}$$

Here p and q are considered modulo N. Note that this matches the generic formula with $\hbar = \frac{2\pi}{N}$ and $s = \hbar\varsigma$. It may also be constructed by composing the natural quotient

$$\mathcal{H} = \mathbb{Z}^{g+1} \rtimes \mathbb{Z}^g \longrightarrow\!\!\!\!\!\!\rightarrow \left(\mathbb{Z}/2N \times (\mathbb{Z}/N)^g\right) \rtimes (\mathbb{Z}/N)^g$$

with the representation of the right-hand group induced from the one-dimensional representation $\mathbb{Z}/2N \times (\mathbb{Z}/N)^g \twoheadrightarrow \mathbb{Z}/2N \hookrightarrow \mathbb{S}^1$, where the second map is $t \mapsto \exp\left(\frac{\pi i t}{N}\right)$. The previous construction applies. We get a finite dimensional Weil projective representation of the symplectic group which linearises on the metaplectic group for even N, while for odd N it can be linearised on the symplectic group itself. The analogue of Theorem 56 in this finite dimensional case produces a representation of the native mapping class group $\mathfrak{M}(\Sigma)$ for odd N, and a representation of the metaplectic mapping class group $\widetilde{\mathfrak{M}}(\Sigma)$ for even N. As exposed in [19], the Weil representation can be realised geometrically by theta functions and can also be interpreted and extended as a $U(1)$ TQFT. An alternative exposition can be found in [18]; see e.g. the statement for the resolution of the projective ambiguity in Chapter 3, Theorem 4.1.

7 Relation to the Moriyama and Magnus representations

7.1 The Moriyama representation.

Moriyama [31] studied the action of the mapping class group $\mathfrak{M}(\Sigma)$ on the homology group $H_n^{BM}(\mathcal{F}_n(\Sigma'); \mathbb{Z})$ with trivial coefficients, where Σ' denotes $\Sigma = \Sigma_{g,1}$ minus a point on its boundary and $\mathcal{F}_n(-)$ denotes the ordered configuration space. On the other hand, our construction (30) (Theorem 39) may be re-interpreted as a twisted representation

$$\mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}^{\mathrm{tw}}_{\mathbb{Z}[\mathcal{H}]}\left(H_n^{BM}(\mathcal{C}_n(\Sigma'); \mathbb{Z}[\mathcal{H}])\right). \tag{49}$$

We pause to explain this re-interpretation. We must first of all explain the twisted automorphism group on the right-hand side of (49). Let us write $\mathrm{Mod}_\bullet$ for the category whose objects are pairs (R, M) of a ring R and a right R-module M, and whose morphisms are pairs $(\theta\colon R \to R', \varphi\colon M \to M')$ such that $\varphi(mr) = \varphi(m)\theta(r)$. The automorphism group of (R, M) in $\mathrm{Mod}_\bullet$ is written $\mathrm{Aut}^{\mathrm{tw}}_R(M)$; note that this is generally larger than the automorphism group $\mathrm{Aut}_R(M)$ of M in Mod_R.

If we set $V = \mathbb{Z}[\mathcal{H}]$, then (30) is a functor of the form $\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}) \to \mathrm{Mod}_{\mathbb{Z}[\mathcal{H}]}$. But any functor of the form $\mathrm{Ac}(G \curvearrowright K) \to \mathrm{Mod}_{\mathbb{Z}[K]}$ corresponds to a homomorphism $G \to \mathrm{Aut}^{\mathrm{tw}}_{\mathbb{Z}[K]}(M)$, where the $\mathbb{Z}[K]$-module M is the image of the object $\mathrm{id} \in \mathrm{Ac}(G \curvearrowright K)$.

(Compare Remark 37, which describes the reverse procedure.) Thus (30) corresponds to a homomorphism

$$\mathfrak{M}(\Sigma) \longrightarrow \mathrm{Aut}^{\mathrm{tw}}_{\mathbb{Z}[\mathcal{H}]}\Big(H_n^{BM}(\mathcal{C}_n(\Sigma),\mathcal{C}_n(\Sigma,\partial^-(\Sigma));\mathbb{Z}[\mathcal{H}])\Big).$$

Finally, removing a point (equivalently, removing the closed interval $\partial^-(\Sigma)$) from the boundary of Σ corresponds, on Borel-Moore homology of configuration spaces $\mathcal{C}_n(\Sigma)$, to taking homology relative to the subspace $\mathcal{C}_n(\Sigma,\partial^-(\Sigma))$ of configurations having at least one point in the interval. Thus $H_n^{BM}(\mathcal{C}_n(\Sigma),\mathcal{C}_n(\Sigma,\partial^-(\Sigma));\mathbb{Z}[\mathcal{H}])$ and $H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[\mathcal{H}])$ are isomorphic as $\mathbb{Z}[\mathcal{H}]$-modules, and we obtain (49).

When $n = 2$, Moriyama's representation is a quotient of ours: there is a quotient of groups $\mathcal{H} \twoheadrightarrow \mathbb{Z}/2 = \Sigma_2$ given by sending $\sigma \mapsto \sigma$ and $a_i, b_i \mapsto 1$, which induces a quotient of twisted $\mathfrak{M}(\Sigma)$-representations

$$H_2^{BM}(\mathcal{C}_2(\Sigma');\mathbb{Z}[\mathcal{H}]) \longrightarrow\!\!\!\!\!\rightarrow H_2^{BM}(\mathcal{C}_2(\Sigma');\mathbb{Z}[\Sigma_2]) \cong H_2^{BM}(\mathcal{F}_2(\Sigma');\mathbb{Z}), \tag{50}$$

where the isomorphism on the right-hand side follows from Shapiro's lemma. (Shapiro's lemma holds for arbitrary coverings with ordinary homology, and for *finite* coverings with Borel-Moore homology.) It follows that the kernel of our representation is a subgroup of the kernel of $H_2^{BM}(\mathcal{F}_2(\Sigma');\mathbb{Z})$, which was proven by Moriyama to be the Johnson kernel $\mathfrak{J}(2)$. In §8 we will compute the action of a genus-1 separating twist $T_\gamma \in \mathfrak{J}(2)$ on $H_2^{BM}(\mathcal{C}_2(\Sigma');\mathbb{Z}[\mathcal{H}])$, and in particular show that it is (very) non-trivial; see Theorem 66. Thus the kernel of $H_2^{BM}(\mathcal{C}_2(\Sigma');\mathbb{Z}[\mathcal{H}])$ is strictly smaller than $\mathfrak{J}(2)$.

For any $n \geq 2$, we have a quotient of twisted $\mathfrak{M}(\Sigma)$-representations

$$H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[\mathcal{H}]) \longrightarrow\!\!\!\!\!\rightarrow H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}).$$

By Shapiro's lemma and the universal coefficient theorem (together with the fact that the integral Borel-Moore homology of $\mathcal{C}_n(\Sigma')$ is free abelian), there are isomorphisms

$$H_n^{BM}(\mathcal{F}_n(\Sigma');\mathbb{Z}) \cong H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[\Sigma_n]) \cong H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}) \otimes \mathbb{Z}[\Sigma_n].$$

Thus the kernel of the representation $H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z})$ is the same as the kernel of the representation $H_n^{BM}(\mathcal{F}_n(\Sigma');\mathbb{Z})$, since $\mathfrak{M}(\Sigma)$ acts trivially on Σ_n. (This is also shown in [32].) The latter kernel was proven by Moriyama to be the nth term $\mathfrak{J}(n)$ of the Johnson filtration.

Summarising this discussion, we have:

Proposition 58. *The kernel of the twisted $\mathfrak{M}(\Sigma)$-representation (49) is contained in the nth term $\mathfrak{J}(n)$ of the Johnson filtration. When $n = 2$ it is moreover a proper subgroup of the Johnson kernel $\mathfrak{J}(2)$.*

7.2 The Magnus representation.

The kernel of our representation (49) is also contained in the kernel of the Magnus representation. This may be seen as follows. The $\mathfrak{M}(\Sigma)$-equivariant surjection $\mathcal{H} \twoheadrightarrow H$ induces a quotient of twisted $\mathfrak{M}(\Sigma)$-representations

$$H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[\mathcal{H}]) \longrightarrow\!\!\!\!\!\rightarrow H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[H]). \tag{51}$$

By a similar argument as above, the kernel of the representation $H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[H])$ is the same as the kernel of the representation $H_n^{BM}(\mathcal{F}_n(\Sigma');\mathbb{Z}[H])$. Moreover, it is shown in [32] that there is an inclusion of $\mathfrak{M}(\Sigma)$-representations

$$\big[H_1^{BM}(\mathcal{F}_1(\Sigma');\mathbb{Z}[H])\big]^{\otimes n} \longhookrightarrow H_n^{BM}(\mathcal{F}_n(\Sigma');\mathbb{Z}[H]), \tag{52}$$

where $H_1^{BM}(\mathcal{F}_1(\Sigma');\mathbb{Z}[H])$ is the *Magnus representation* of $\mathfrak{M}(\Sigma)$. The maps of representations (51) and (52) imply that

$$\begin{aligned}\ker\big[H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[\mathcal{H}])\big] \;\subseteq \ker\big[H_n^{BM}(\mathcal{C}_n(\Sigma');\mathbb{Z}[H])\big] \\ = \;\ker\big[H_n^{BM}(\mathcal{F}_n(\Sigma');\mathbb{Z}[H])\big] \;\subseteq\; \ker(\mathrm{Magnus}).\end{aligned}$$

Combining this with Proposition 58, we have:

Proposition 59. *The kernel of* (49) *is contained in* $\mathfrak{J}(n) \cap \ker(\mathrm{Magnus})$.

It is known [35, §6] that the kernel of the Magnus representation does not contain $\mathfrak{J}(n)$ for any $n \geq 1$, so this implies that the kernel of (49) is strictly contained in $\mathfrak{J}(n)$.

7.3 Other related representations.

Recently, the representations of $\mathfrak{M}(\Sigma)$ on the ordinary (rather than Borel-Moore) homology of the configuration space $\mathcal{F}_n(\Sigma)$ has been studied[2] by Bianchi, Miller and Wilson [9]: they prove that, for each n and i, the kernel of the $\mathfrak{M}(\Sigma)$-representation $H_i(\mathcal{F}_n(\Sigma);\mathbb{Z})$ contains $\mathfrak{J}(i)$, and is in general strictly *larger* than $\mathfrak{J}(i)$. They conjecture that the kernel of the $\mathfrak{M}(\Sigma)$-representation on the total homology $H_*(\mathcal{F}_n(\Sigma);\mathbb{Z})$ is equal to the subgroup generated by $\mathfrak{J}(n)$ and the Dehn twist around the boundary.

The $\mathfrak{M}(\Sigma)$-representation $H_i(\mathcal{C}_n(\Sigma);\mathbb{F})$ for certain field coefficients $\mathbb{F}$ has been completely computed. For $\mathbb{F} = \mathbb{F}_2$ it has been computed in [8, Theorem 3.2] and is *symplectic*, i.e. it restricts to the trivial action on the Torelli group $\mathfrak{T}(\Sigma) = \mathfrak{J}(1)$. For $\mathbb{F} = \mathbb{Q}$ it has been computed in [34, Theorem 1.4] and is not symplectic, but it restricts to the trivial action on the Johnson kernel $\mathfrak{J}(2)$.

8 Computations for $n = 2$

In this section we will do some computations in the case $n = 2$, when V is the regular representation $\mathbb{Z}[\mathcal{H}]$ of the Heisenberg group $\mathcal{H}$. The main goal is to obtain in this case an explicit formula for the action of a Dehn twist along a genus 1 separating curve. When the surface has genus 1 this is displayed in Figure 8; in general, the formula is given by Theorem 66.

We will start with the case where the surface itself has genus 1, where we first compute the action of the Dehn twists T_a, T_b, along the standard essential curves a, b. Since T_a and

[2] This is equivalent to studying the homology of $\mathcal{F}_n(\Sigma')$ since the inclusion $\mathcal{F}_n(\Sigma') \hookrightarrow \mathcal{F}_n(\Sigma)$ is a homotopy equivalence. On the other hand, for *Borel-Moore* homology, this would not be equivalent, since the inclusion is not a *proper* homotopy equivalence.

T_b act non-trivially on the local system $\mathbb{Z}[\mathcal{H}]$, they do not act by automorphisms, but give isomorphisms in the category of spaces with local systems, which, after taking homology with local coefficients, give isomorphisms in the category of $\mathbb{Z}[\mathcal{H}]$-modules. We refer to [17, Chapter 5] for functoriality results concerning homology with local coefficients. The upshot is a twisted action of the full mapping class group $\mathfrak{M}(\Sigma)$. As described in §5.1, a *twisted action* (over a ring R) of a group G is a functor $\mathrm{Ac}(G \curvearrowright X) \to \mathrm{Mod}_R$, where $\mathrm{Ac}(G \curvearrowright X)$ is the *action groupoid* associated to an action of G on some set X. In the present setting, we have $G = \mathfrak{M}(\Sigma)$, $X = \mathcal{H}$ and $R = \mathbb{Z}[\mathcal{H}]$, so the twisted representation is of the form

$$\mathrm{Ac}(\mathfrak{M}(\Sigma) \curvearrowright \mathcal{H}) \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathcal{H}]}. \tag{53}$$

We briefly recall from §5.2 some of the relevant details of the construction of this twisted representation. Let $f \in \mathfrak{M}(\Sigma)$ and let $f_{\mathcal{H}}$ be its action on the Heisenberg group. Then the Heisenberg homology $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})$ is defined from the regular covering space $\widetilde{\mathcal{C}}_n(\Sigma)$ associated with the quotient $\phi\colon \mathbb{B}_n(\Sigma) \twoheadrightarrow \mathcal{H}$. As explained in §5.2, at the level of homology there is a twisted functoriality and, in particular, associated with f, we get a right $\mathbb{Z}[\mathcal{H}]$-linear isomorphism

$$\mathcal{C}_n(f)_*\colon H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_{f_{\mathcal{H}}^{-1}} \longrightarrow H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})\ .$$

Our choice for twisting on the source with $f_{\mathcal{H}}^{-1}$ rather than on the target with $f_{\mathcal{H}}$ will slightly simplify the writing of the matrix. Note also that when working with coefficients in a left $\mathbb{Z}[\mathcal{H}]$-representation V the twisting on the right by $f_{\mathcal{H}}^{-1}$ will correspond to twisting the action on V by $f_{\mathcal{H}}$. More generally, for any $\tau \in \mathrm{Aut}(\mathcal{H})$, we have a *shifted* isomorphism

$$(\mathcal{C}_n(f)_*)_\tau\colon H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_{f_{\mathcal{H}}^{-1} \circ \tau} \longrightarrow H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); \mathcal{H})_\tau\ .$$

In terms of the functor (53), the above map $(\mathcal{C}_n(f)_*)_\tau$ is the image of the morphism $f\colon f_{\mathcal{H}}^{-1} \circ \tau \to \tau$. If f, g are two mapping classes, the composition formula (functoriality of (53)) states the following:

$$\mathcal{C}_n(g \circ f)_* = \mathcal{C}_n(g)_* \circ (\mathcal{C}_n(f)_*)_{g_{\mathcal{H}}^{-1}}\ .$$

We will need to compute compositions in specific bases. Note that a basis B for a right $\mathbb{Z}[\mathcal{H}]$-module M is also a basis for the twisted module M_τ, $\tau \in \mathrm{Aut}(\mathcal{H})$.

Lemma 60. *Let M, M' be free right $\mathbb{Z}[\mathcal{H}]$-modules with fixed bases B, B' and let $\tau \in \mathrm{Aut}(\mathcal{H})$. If a $\mathbb{Z}[\mathcal{H}]$-linear map $F : M \to M'$ has matrix $\mathrm{Mat}(F)$ in the bases B, B', then the matrix of the shifted $\mathbb{Z}[\mathcal{H}]$-linear map $F_\tau : M_\tau \to M'_\tau$ is $\tau^{-1}(\mathrm{Mat}(F))$.*

Here the action of τ^{-1} on the matrix is the action on all individual coefficients.
Proof. We note that the maps F and F_τ are equal as maps of $\mathbb{Z}$-modules. Let $B = (e_j)_{j\in J}$, $B' = (f_i)_{i\in I}$, $\mathrm{Mat}(F) = (m_{i,j})_{i\in I, j\in J}$. Then for coefficients $h_j \in \mathcal{H}$, $j \in J$, we

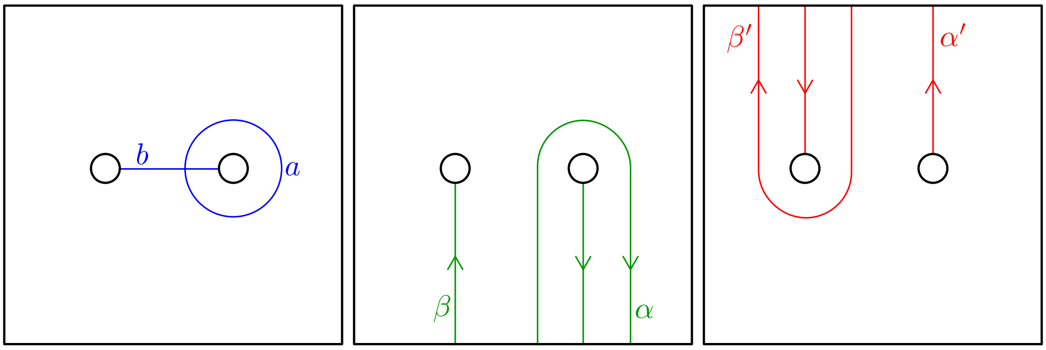


Figure 5: The closed curves a, b and the arcs α, β, α', β'.

have

$$
\begin{aligned}
F_\tau\left(\sum_j e_j \cdot_\tau h_j\right) &= F\left(\sum_j e_j \tau(h_j)\right) \\
&= \sum_{i,j} f_i m_{ij} \tau(h_j) \\
&= \sum_{i,j} f_i \cdot_\tau \tau^{-1}(m_{ij}) h_j,
\end{aligned}
$$

which gives the stated result. ■

8.1 Genus one

Here we consider the genus 1 case with $n = 2$ configuration points. Let a, b be simple closed curves representing the symplectic basis of $H_1(\Sigma)$ previously denoted (a_1, b_1). We will use the same notation a, b for the curves, their homology classes and their lifts in $\mathcal{H}$ which were previously $\tilde{a}$, $\tilde{b}$. The corresponding Dehn twists are denoted by T_a, T_b. The homology $H_2^{BM}(\mathcal{C}_2(\Sigma), \mathcal{C}_2(\Sigma, \partial^-(\Sigma)); \mathcal{H})$ is a free module of rank 3 over $\mathbb{Z}[\mathcal{H}]$. A basis was described in Theorem 10. Here we replace γ_1, γ_2 by α, β depicted in Figure 5, and the basis is denoted by $w(\alpha) = E_{(2,0)}$, $w(\beta) = E_{(0,2)}$, $v(\alpha, \beta) = E_{(1,1)}$. In more detail, $w(\alpha)$ is represented by the cycle in the 2-point configuration space given by the subspace where both points lie on the arc α. Similarly, $w(\beta)$ is given by the subspace where both points lie on β and $v(\alpha, \beta)$ is given by the subspace where exactly one point lies on each of these arcs.

In fact, we have to be even more careful to specify these elements precisely, since the preceding description only determines them *up to the action of the deck transformation group* $\mathcal{H}$, because we have just described cycles in the configuration space $\mathcal{C}_2(\Sigma)$, whereas cycles for the Heisenberg-twisted homology are cycles in the covering space $\widetilde{\mathcal{C}}_2(\Sigma)$. To specify such a lifting of the cycles in $\mathcal{C}_2(\Sigma)$ that we have described, we first choose once and for all a base configuration c_0 contained in $\partial\Sigma$ and a lift of c_0 to $\widetilde{\mathcal{C}}_2(\Sigma)$. A lift of a

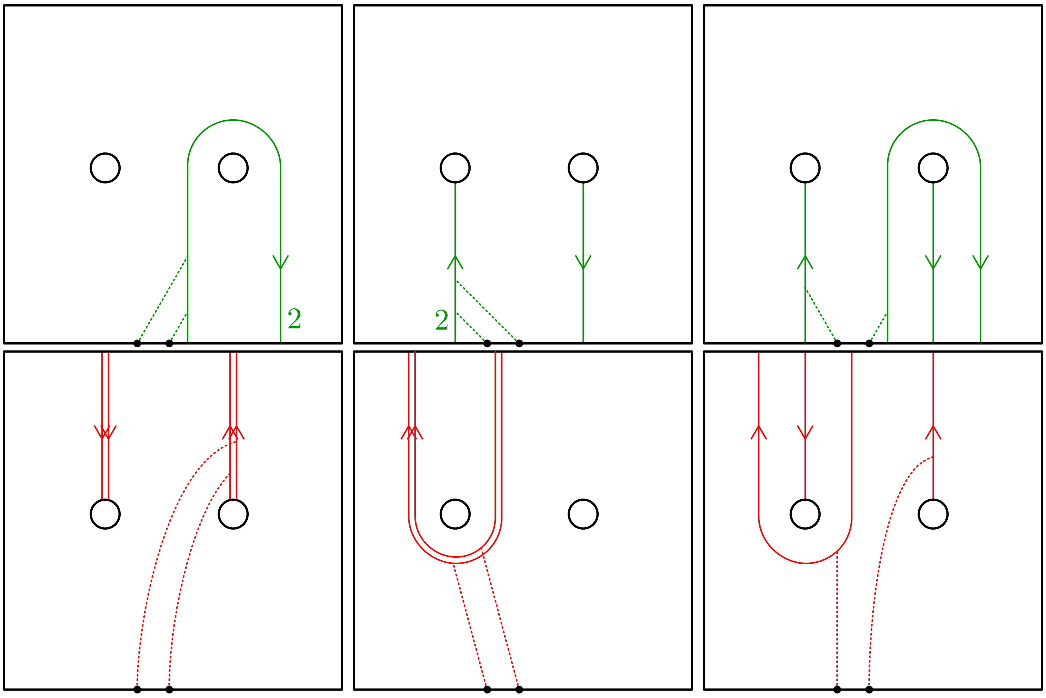


Figure 6: Tethers.

cycle to $\widetilde{\mathcal{C}}_2(\Sigma)$ is therefore determined by a choice of a path (called a "tether") in $\mathcal{C}_2(\Sigma)$ from a point in the cycle to c_0. For $w(\alpha)$, $w(\beta)$ and $v(\alpha,\beta)$, we choose these tethers as illustrated in the top row of Figure 6.

By Poincaré duality, and the fact that $\mathcal{C}_2(\Sigma)$ is a connected, oriented 4-manifold with boundary $\mathcal{C}_2(\Sigma,\partial\Sigma) = \{c \in \mathcal{C}_2(\Sigma) \mid c \cap \partial\Sigma \neq \varnothing\}$, we have a non-degenerate pairing

$$\langle -,- \rangle \colon H_2^{BM}(\mathcal{C}_2(\Sigma),\partial^-;\mathbb{Z}[\mathcal{H}]) \otimes H_2(\mathcal{C}_2(\Sigma),\partial^+;\mathbb{Z}[\mathcal{H}]) \longrightarrow \mathbb{Z}[\mathcal{H}], \tag{54}$$

where $\partial^\pm$ is an abbreviation of $\mathcal{C}_2(\Sigma,\partial^\pm(\Sigma))$, and we note that the boundary $\partial\mathcal{C}_2(\Sigma) = \mathcal{C}_2(\Sigma,\partial\Sigma)$ decomposes as $\partial^+ \cup \partial^-$, corresponding to the decomposition of the boundary of the surface $\partial\Sigma = \partial^+(\Sigma) \cup \partial^-(\Sigma)$. (Formally, it is a *manifold triad.*) There are natural elements of $H_2(\mathcal{C}_2(\Sigma),\partial^+;\mathbb{Z}[\mathcal{H}])$ that are dual to $w(\alpha)$, $w(\beta)$ and $v(\alpha,\beta)$ with respect to this pairing, which we denote by $\overline{w}(\alpha')$, $\overline{w}(\beta')$ and $v(\alpha',\beta')$ respectively. The element $v(\alpha',\beta')$ is defined exactly as above: it is given by the subspace of 2-point configurations where one point lies on each of the arcs α' and β' of Figure 5. The element $\overline{w}(\alpha')$ is defined as follows: first replace the arc α' with two parallel copies α'_1 and α'_2 (as in the bottom-left of Figure 6), and then $\overline{w}(\alpha')$ is given by the subspace of 2-point configurations where one point lies on each of α'_1 and α'_2. The element $\overline{w}(\beta')$ is defined exactly analogously. Again, in order to specify these elements precisely, we have to choose tethers, which are illustrated in the bottom row of Figure 6.

A practical description of the pairing (54) is as follows. Let $x = w(\gamma)$ or $v(\gamma,\delta)$ for disjoint arcs γ, δ with endpoints on $\partial^-(\Sigma)$, and choose a tether for x, namely a path

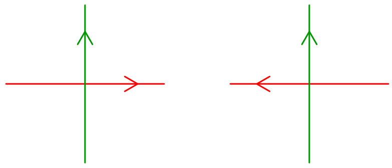


Figure 7: Sign convention for intersections between cycles representing elements of the homology groups $H_n^{BM}(\mathcal{C}_n(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])$ and $H_n(\mathcal{C}_n(\Sigma), \partial^+; \mathbb{Z}[\mathcal{H}])$.

t_x from c_0 to a point in x. Similarly, let $y = \overline{w}(\epsilon)$ or $v(\epsilon, \zeta)$ for disjoint arcs ϵ, ζ with endpoints on $\partial^+(\Sigma)$, and choose a tether t_y for y. Suppose that the arcs $\gamma \sqcup \delta$ intersect the arcs $\epsilon \sqcup \zeta$ transversely. Then the pairing (54) is given by the formula

$$\langle [x, t_x], [y, t_y] \rangle = \sum_{p=\{p_1,p_2\} \in x \cap y} \mathrm{sgn}(p_1).\mathrm{sgn}(p_2).\mathrm{sgn}(\ell_p).\phi(\ell_p), \tag{55}$$

where $\ell_p \in \mathbb{B}_2(\Sigma)$ is the loop in $\mathcal{C}_2(\Sigma)$ given by concatenating:

- the tether t_x from c_0 to a point in x,
- a path in x to the intersection point p,
- a path in y from p to the endpoint of the tether t_y,
- the reverse of the tether t_y back to c_0,

$\mathrm{sgn}(\ell_p) \in \{+1, -1\}$ is the sign of the induced permutation in $\mathfrak{S}_2$ and $\mathrm{sgn}(p_i) \in \{+1, -1\}$ is given by the sign convention in Figure 7. (In fact, there should be an extra global -1 sign on the right-hand side of (55), which we have suppressed for simplicity. Thus (55) is really a formula for $-$(54). This global sign ambiguity does not affect our calculations, since all we need is *a* non-degenerate pairing of the form (54), and any non-degenerate pairing multiplied by a unit is again a non-degenerate pairing. This extra global sign also appears in Bigelow's formula [11, page 475, ten lines above Lemma 2.1]. See Appendix C for further explanations of these signs.)

With this description of (54), it is easy to verify that the matrix

$$\begin{pmatrix} \langle [w(\alpha)], [\overline{w}(\alpha')] \rangle & \langle [w(\alpha)], [\overline{w}(\beta')] \rangle & \langle [w(\alpha)], [v(\alpha', \beta')] \rangle \\ \langle [w(\beta)], [\overline{w}(\alpha')] \rangle & \langle [w(\beta)], [\overline{w}(\beta')] \rangle & \langle [w(\beta)], [v(\alpha', \beta')] \rangle \\ \langle [v(\alpha,\beta)], [\overline{w}(\alpha')] \rangle & \langle [v(\alpha,\beta)], [\overline{w}(\beta')] \rangle & \langle [v(\alpha,\beta)], [v(\alpha', \beta')] \rangle \end{pmatrix} \in \mathrm{Mat}_{3,3}(\mathbb{Z}[\mathcal{H}])$$

is the identity; this is the precise sense in which these two 3-tuples of elements are "dual" to each other.[3]

Theorem 61. *With respect to the ordered basis* $(w(\alpha), w(\beta), v(\alpha, \beta))$*:*

(a) *The matrix for the isomorphism*

$$\mathcal{T}_a = \mathcal{C}_2(T_a)_* \colon H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])_{(T_a^{-1})_{\mathcal{H}}} \longrightarrow H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])$$

[3] Since we know that $w(\alpha)$, $w(\beta)$ and $v(\alpha, \beta)$ form a basis for the $\mathbb{Z}[\mathcal{H}]$-module $H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])$, it follows that the elements $\overline{w}(\alpha')$, $\overline{w}(\beta')$ and $v(\alpha', \beta')$ are $\mathbb{Z}[\mathcal{H}]$-linearly independent in the $\mathbb{Z}[\mathcal{H}]$-module $H_2(\mathcal{C}_2(\Sigma), \partial^+; \mathbb{Z}[\mathcal{H}])$, although they do not necessarily span it.

is

$$M_a = \begin{pmatrix} 1 & u^2a^{-2}b^2 & (u^{-1}-1)a^{-1}b \\ 0 & 1 & 0 \\ 0 & -a^{-1}b & 1 \end{pmatrix} .$$

(b) *The matrix for the isomorphism*

$$\mathcal{T}_b = \mathcal{C}_2(T_b)_* \colon H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])_{(T_b^{-1})_{\mathcal{H}}} \longrightarrow H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])$$

is

$$M_b = \begin{pmatrix} u^{-2}b^2 & 0 & 0 \\ -u^{-1} & 1 & 1-u^{-1} \\ -u^{-1}b & 0 & b \end{pmatrix} .$$

Proof. Let us simplify the notation for the basis and the corresponding dual homology classes by

$$(e_1, e_2, e_3) = (w(\alpha), w(\beta), v(\alpha, \beta)) \qquad (e_1', e_2', e_3') = (\overline{w}(\alpha'), \overline{w}(\beta'), v(\alpha', \beta')).$$

Using the non-degenerate pairing (54) and elementary linear algebra, we have that

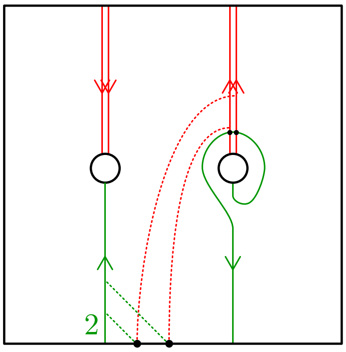
$$\mathcal{C}_2(f)_*(e_i) = \sum_{j=1}^{3} e_j . \langle \mathcal{C}_2(f)_*(e_i), e_j' \rangle$$

for any $f \in \mathfrak{M}(\Sigma)$. Computing the matrices M_a and M_b therefore consists in computing $\langle \mathcal{T}_a(e_i), e_j' \rangle$ and $\langle \mathcal{T}_b(e_i), e_j' \rangle$ for $i, j \in \{1, 2, 3\}$. We will explain how to compute two of these 18 elements of $\mathbb{Z}[\mathcal{H}]$, the remaining 16 being left as exercises for the reader. In each case the idea is the same: apply the Dehn twist to the explicit cycle (described above) representing the homology class e_i, and then use the formula (55) to compute the pairing.

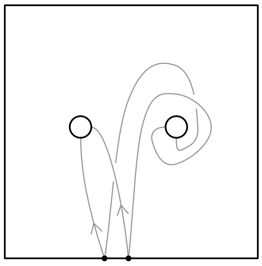
We begin by computing $\langle \mathcal{T}_a(e_2), e'_1\rangle = \langle \mathcal{T}_a(w(\beta)), \overline{w}(\alpha')\rangle$, the top-middle entry of M_a.

$$\langle \mathcal{T}_a(w(\beta)), \overline{w}(\alpha')\rangle = \langle w(T_a(\beta)), \overline{w}(\alpha')\rangle$$

$=$



$= (-1).(-1).(+1).\phi$

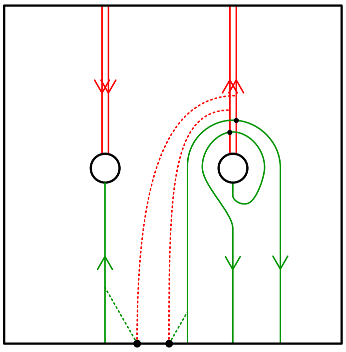
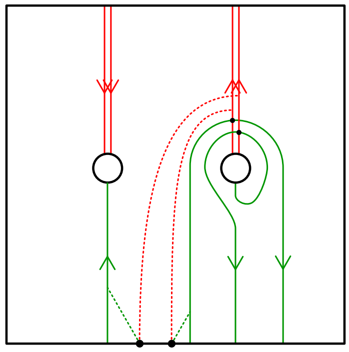


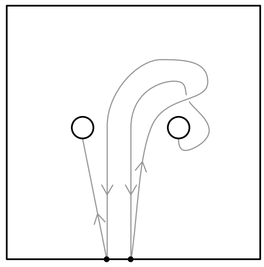
$$= \phi(a^{-1}b\sigma^{-1}a^{-1}b\sigma)$$
$$= a^{-1}ba^{-1}b$$
$$= u^2a^{-2}b^2.$$

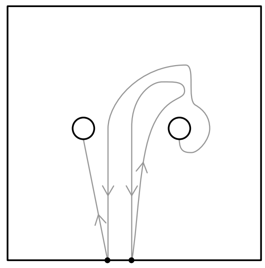
We next calculate $\langle \mathcal{T}_a(e_3), e'_1\rangle = \langle \mathcal{T}_a(v(\alpha,\beta)), \overline{w}(\alpha')\rangle$, the top-right entry of M_a. This is slightly more complicated, since in this case there are two intersection points in the configuration space $\mathcal{C}_2(\Sigma)$, so we obtain a Heisenberg polynomial (i.e. element of $\mathbb{Z}[\mathcal{H}]$)

with two terms.

$$\langle \mathcal{T}_a(v(\alpha,\beta)), \overline{w}(\alpha')\rangle = \langle v(\alpha, T_a(\beta)), \overline{w}(\alpha')\rangle$$

$=$  $+$ 

$= (-1).(+1).(-1).\phi \Big($  $\Big)$

$+ (+1).(-1).(+1).\phi \Big($  $\Big)$

$$= \phi(\sigma^{-1}a^{-1}b) - \phi(a^{-1}b)$$
$$= u^{-1}a^{-1}b - a^{-1}b$$
$$= (u^{-1} - 1)a^{-1}b.$$

The other 16 entries of the matrices M_a and M_b may be computed analogously. ■

Notation 62. To shorten the notation in the following, we will use the abbreviation

$$A := H_2^{BM}(\mathcal{C}_2(\Sigma), \mathcal{C}_2(\Sigma, \partial^-(\Sigma)); \mathcal{H}) = H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}]).$$

Remark 63 (*Verifying the braid relation.*)**.** Recall that $\mathfrak{M}(\Sigma_{1,1})$ is generated by T_a and T_b subject to the single relation $T_aT_bT_a = T_bT_aT_b$. It must therefore be the case that the isomorphism

$$A_{(T_aT_bT_a)^{-1}_{\mathcal{H}}} \xrightarrow{(\mathcal{T}_a)_{(T_aT_b)^{-1}_{\mathcal{H}}}} A_{(T_aT_b)^{-1}_{\mathcal{H}}} \xrightarrow{(\mathcal{T}_b)_{(T_a)^{-1}_{\mathcal{H}}}} A_{(T_a)^{-1}_{\mathcal{H}}} \xrightarrow{\mathcal{T}_a} A$$

is equal to the isomorphism

$$A_{(T_bT_aT_b)^{-1}_{\mathcal{H}}} \xrightarrow{(\mathcal{T}_b)_{(T_bT_a)^{-1}_{\mathcal{H}}}} A_{(T_bT_a)^{-1}_{\mathcal{H}}} \xrightarrow{(\mathcal{T}_a)_{(T_b)^{-1}_{\mathcal{H}}}} A_{(T_b)^{-1}_{\mathcal{H}}} \xrightarrow{\mathcal{T}_b} A$$

in other words, using Lemma 60, we must have the following equality of matrices:

$$M_a.(T_a)_{\mathcal{H}}(M_b).(T_aT_b)_{\mathcal{H}}(M_a) = M_b.(T_b)_{\mathcal{H}}(M_a).(T_bT_a)_{\mathcal{H}}(M_b), \tag{56}$$

where M_a and M_b are as in Theorem 61 and the automorphisms $(T_a)_{\mathcal{H}}, (T_b)_{\mathcal{H}} \in \mathrm{Aut}(\mathcal{H})$ are extended linearly to automorphisms of $\mathbb{Z}[\mathcal{H}]$ and thus to automorphisms of matrices over $\mathbb{Z}[\mathcal{H}]$. Indeed, one may calculate that both sides of (56) are equal to

$$\begin{pmatrix} 0 & u^2a^{-2}b^2 & 0 \\ -u^{-1} & 1+(u^{-3}-u^{-2})a^{-1}-u^{-5}a^{-2} & (1-u^{-1})(1+u^{-3}a^{-1}) \\ 0 & -a^{-1}b-u^{-1}a^{-2}b & u^{-1}a^{-1}b \end{pmatrix}. \tag{57}$$

Remark 64 (*The Dehn twist around the boundary.*)**.** In a similar way, we may compute the matrix M_∂ for the action $\mathcal{T}_\partial$ of the Dehn twist T_∂ around the boundary of $\Sigma_{1,1}$. We note that T_∂ lies in the Chillingworth subgroup of $\mathfrak{M}(\Sigma_{1,1})$, so its action on $\mathcal{H}$ is trivial and the action $\mathcal{T}_\partial$ is an *automorphism*

$$\mathcal{T}_\partial \colon A \longrightarrow A.$$

However, to compute its matrix M_∂, it is convenient to decompose $\mathcal{T}_\partial$ into isomorphisms as follows. Write $g = T_aT_bT_a = T_bT_aT_b$, so that $T_\partial = g^4$. Then $\mathcal{T}_\partial$ decomposes as

$$A = A_{g^{-4}_{\mathcal{H}}} \xrightarrow{(\mathcal{T}_g)_{g^{-3}_{\mathcal{H}}}} A_{g^{-3}_{\mathcal{H}}} \xrightarrow{(\mathcal{T}_g)_{g^{-2}_{\mathcal{H}}}} A_{g^{-2}_{\mathcal{H}}} \xrightarrow{(\mathcal{T}_g)_{g^{-1}_{\mathcal{H}}}} A_{g^{-1}_{\mathcal{H}}} \xrightarrow{\mathcal{T}_g} A$$

where $\mathcal{T}_g$ denotes the action of g, given by the matrix (57) above. The matrix M_∂ may therefore be obtained by multiplying together four copies of (57), shifted by the actions of id, $g_{\mathcal{H}}$, $g^2_{\mathcal{H}}$ and $g^3_{\mathcal{H}}$ respectively. This may be implemented in Sage to show that M_∂ is equal to the matrix displayed in Figure 8. More details of these Sage calculations are given in Appendix D.

One may verify explicitly by hand that, if we set $a = b = u^2 = 1$ in the matrix $M_\partial =$ (Figure 8), it simplifies to the identity matrix. This is expected, since applying this specialisation to our representation recovers the second Moriyama representation (as discussed in §7; see in particular the quotient (50) of $\mathfrak{M}(\Sigma)$-representations), whose kernel is the Johnson kernel $\mathfrak{J}(2)$ by [31], which contains T_∂.

8.2 Higher genus

For arbitrary genus $g \geq 1$, we view the surface $\Sigma = \Sigma_{g,1}$ as the quotient of the punctured rectangle depicted in Figure 9, where the $2g$ holes are identified in pairs by reflection.

$$
\begin{pmatrix}
\begin{matrix} u^{-8}b^2+u^{-4}a^{-2}-ua^{-2}b^2+(u^{-1}-u^{-2})a^{-2}b+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(u^{-4}-u^{-5})a^{-1}b \end{matrix} &
\begin{matrix} (u^2+1-2u^{-1}+u^{-2}+u^{-4})a^{-2}b^2-ua^{-2}b^4+ \\ (-u^2+u+u^{-1}-u^{-2})a^{-2}b^3-u^{-3}a^{-2}+ \\ (-1+u^{-1}+u^{-3}-u^{-4})a^{-2}b \end{matrix} &
\begin{matrix} (-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-2}b+ \\ (u-1)a^{-2}b^3+(u^2-u-u^{-1}+2u^{-2}-u^{-3})a^{-2}b^2+ \\ (-u^{-3}+u^{-4})a^{-1}b+(u^{-4}-u^{-5})a^{-1}b^3+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}b^2+ \\ (-u^{-3}+u^{-4})a^{-2} \end{matrix} \\
\begin{matrix} -u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^2+ \\ (u^{-1}-u^{-2}-u^{-4}+u^{-5})a+u^{-6}a^{-2}+ \\ (u^{-3}-u^{-4}-u^{-6}+u^{-7})a^{-1} \end{matrix} &
\begin{matrix} 1+u^{-2}-u^{-3}+u^{-6}+u^{-6}a^{-2}b^2-u^{-1}b^2+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(-1+u^{-1}+u^{-3}-u^{-4})b+ \\ (u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})a^{-1}b-u^{-5}a^{-2}+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}+(u^{-5}-u^{-6})a^{-2}b \end{matrix} &
\begin{matrix} (-u^{-6}+u^{-7})a^{-2}b+ \\ (u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b+ \\ (-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})a^{-1}b+ \\ 1-u^{-1}+u^{-2}-3u^{-3}+2u^{-4}+u^{-6}-u^{-7}+ \\ (-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a^{-1} \\ +(u^{-2}-u^{-3})ab+(-1+u^{-1}+u^{-3}-u^{-4})a+ \\ (-u^{-5}+u^{-6})a^{-2} \end{matrix} \\
\begin{matrix} -u^{-6}ab+(-u^{-3}+u^{-4}-u^{-7})b-u^{-4}+ \\ (u^{-1}-u^{-4}+u^{-5})a^{-1}b+u^{-2}a^{-2}b+ \\ (-u^{-3}+u^{-6})a^{-1}+u^{-5}a^{-2} \end{matrix} &
\begin{matrix} (-1-u^{-2}+2u^{-3}-u^{-6})a^{-1}b+u^{-1}a^{-1}b^3+ \\ u^{-2}a^{-2}b^3+(1-u^{-1}-u^{-3}+u^{-4})a^{-1}b^2+ \\ (u^{-1}-u^{-2}+u^{-5})a^{-2}b^2+(-u^{-1}+u^{-4}-u^{-5})a^{-2}b+ \\ (u^{-2}-u^{-5})a^{-1}-u^{-4}a^{-2} \end{matrix} &
\begin{matrix} u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a^{-1}+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-1}b^2+ \\ (-u^{-2}+u^{-3})a^{-2}b^2+ \\ (-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})a^{-1}b+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-2}b+(-u^{-4}+u^{-5})b^2+ \\ (u^{-2}-u^{-3}-u^{-5}+u^{-6})b+(-u^{-4}+u^{-5})a^{-2} \end{matrix}
\end{pmatrix}
$$

Figure 8: The action of the Dehn twist around the boundary of $\Sigma_{1,1}$.

The arcs α_i, β_i for $i \in \{1, \ldots, g\}$ form a symplectic basis for the first homology of Σ relative to the lower edge of the rectangle. Following Theorem 10, a basis for the free $\mathbb{Z}[\mathcal{H}]$-module $H_2^{BM}(\mathcal{C}_2(\Sigma), \mathcal{C}_2(\Sigma, \partial^-(\Sigma)); \mathcal{H})$ is given by the homology classes represented by the 2-cycles

- $w(\epsilon)$ for $\epsilon \in \{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\}$,
- $v(\delta, \epsilon)$ for $\delta, \epsilon \in \{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\}$ with $\delta < \epsilon$

where we use the ordering $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_g < \beta_g$. Here $w(\epsilon)$ denotes the subspace of configurations where both points lie on ϵ and $v(\delta, \epsilon)$ denotes the subspace of configurations where one point lies on each of δ and ϵ. As in the genus 1 setting, we have to be more careful to specify these elements precisely; this is done by choosing, for each of the 2-cycles listed above, a path (called a "tether") in $\mathcal{C}_2(\Sigma)$ from a point in the cycle to c_0, the base configuration, which is contained in the bottom edge of the rectangle. Note that the space of configurations of two points in the bottom edge of the rectangle is contractible, so it is equivalent to choose a path in $\mathcal{C}_2(\Sigma)$ from a point in the cycle to *any* configuration contained in the bottom edge of the rectangle.

For cycles of the form $w(\epsilon)$, we may choose tethers exactly as in the genus 1 setting: see the top-left and top-middle of Figure 6. For cycles of the form $v(\alpha_i, \beta_i)$, we may also choose tethers exactly as in the genus 1 setting: see the top-right of Figure 6. For other cycles of the form $v(\delta, \epsilon)$, we choose tethers as illustrated in Figure 10.

Exactly as in the genus 1 setting, there is a non-degenerate pairing (54) defined via Poincaré duality for the 4-manifold-with-boundary $\mathcal{C}_2(\Sigma)$. Associated to the collection of arcs α'_i, β'_i illustrated in Figure 9 there are elements of $H_2(\mathcal{C}_2(\Sigma), \partial^+; \mathcal{H})$:

- $\overline{w}(\epsilon)$ for $\epsilon \in \{\alpha'_1, \beta'_1, \alpha'_2, \beta'_2, \ldots, \alpha'_g, \beta'_g\}$,

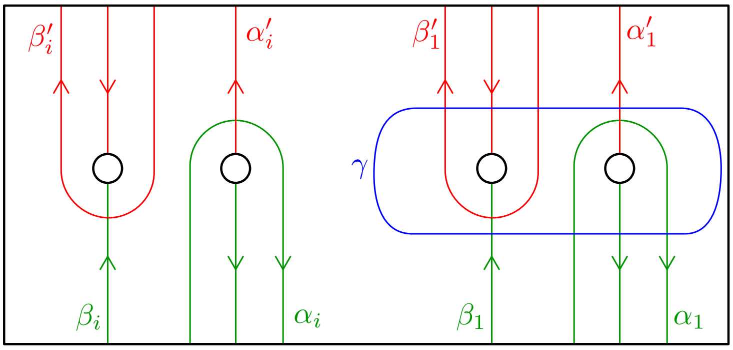


Figure 9: The arcs $\alpha_i, \beta_i, \alpha_i', \beta_i'$ and the closed genus-one-separating curve γ.

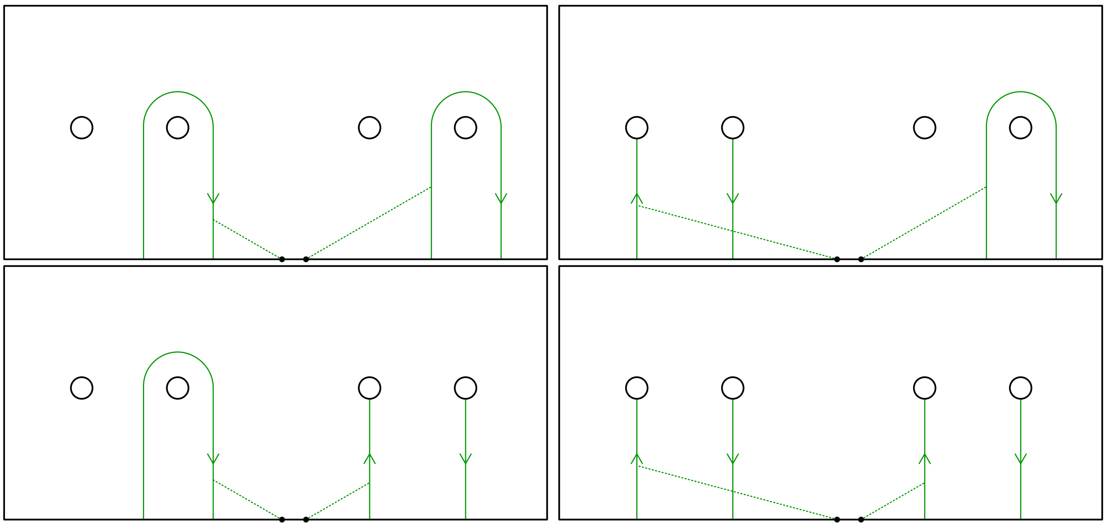


Figure 10: More tethers.

- $v(\delta, \epsilon)$ for $\delta, \epsilon \in \{\alpha_1', \beta_1', \alpha_2', \beta_2', \dots, \alpha_g', \beta_g'\}$ with $\delta < \epsilon$

where we use the ordering $\alpha_1' < \beta_1' < \alpha_2' < \cdots < \alpha_g' < \beta_g'$. Here, $\overline{w}(\epsilon)$ is the subspace of configurations where one point lies on each of ϵ^+ and ϵ^-, where ϵ^+, ϵ^- are two parallel, disjoint copies of ϵ. As above, we specify these elements precisely by choosing tethers (paths in $\mathcal{C}_2(\Sigma)$ from a point on the cycle to a configurations contained in the bottom edge of the rectangle). For elements of the form $\overline{w}(\epsilon)$ or $v(\alpha_i', \beta_i')$, we choose these exactly as in the genus 1 setting; see the bottom row of Figure 6. For other elements of the form $v(\delta, \epsilon)$, we choose them as illustrated in Figure 11.

Remark 65. These choices of tethers may seem a little arbitrary, and indeed they are; however, any different choice would have the effect simply of changing the chosen basis for the Heisenberg homology $H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathcal{H})$ by rescaling each basis vector by a unit of $\mathbb{Z}[\mathcal{H}]$. This would have the effect of conjugating the matrices that we calculate by an invertible diagonal matrix.

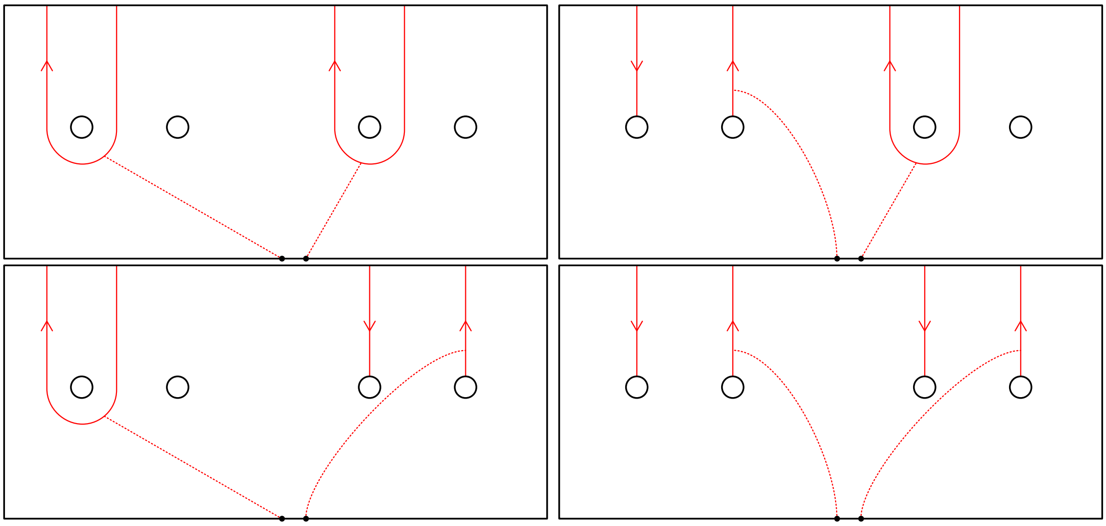


Figure 11: Even more tethers.

The geometric formula (55) for the non-degenerate pairing $\langle - , - \rangle$ holds exactly as in the genus 1 setting, and one may easily verify using this formula that the bases

$$\begin{aligned} \mathcal{B} &= \{w(\epsilon), v(\delta,\epsilon) \mid \delta < \epsilon \in \{\alpha_1, \ldots, \beta_g\}\} \\ \mathcal{B}' &= \{\overline{w}(\epsilon), v(\delta,\epsilon) \mid \delta < \epsilon \in \{\alpha'_1, \ldots, \beta'_g\}\} \end{aligned} \tag{58}$$

for $H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathcal{H})$ and $H_2(\mathcal{C}_2(\Sigma), \partial^+; \mathcal{H})$ respectively are dual with respect to this pairing. Choose a total ordering of $\mathcal{B}$ as follows:

- $w(\alpha_1)$, $w(\beta_1)$, $v(\alpha_1, \beta_1)$,
- $v(\alpha_1, \epsilon)$ for $\epsilon = \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$,
- $v(\beta_1, \epsilon)$ for $\epsilon = \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$,
- followed by all other basis elements in any order,

and similarly for $\mathcal{B}'$. Denote by γ the genus-1 separating curve in $\Sigma = \Sigma_{g,1}$ pictured in Figure 9.

Theorem 66. *With respect to the ordered bases (58), the matrix for the automorphism $\mathcal{T}_\gamma = \mathcal{C}_2(T_\gamma)_*$ of $H_2^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathcal{H})$ is given in block form as*

$$M_\gamma = \begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & p.I & r.I & 0 \\ 0 & q.I & s.I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \tag{59}$$

where Λ is the 3×3 matrix depicted in Figure 8, the middle two columns and rows each have width/height $2g-2$ and the Heisenberg polynomials $p, q, r, s \in \mathbb{Z}[\mathcal{H}]$ are:

- $p = -a^{-1}b + u^{-2}b + u^{-2}a^{-1}$,
- $q = 1 - a + u^{-2} - u^{-2}a^{-1}$,

- $r = a^{-1}(-b + b^2 + u^{-2} - u^{-2}b)$,
- $s = 1 - b + u^{-2} + u^{-2}a^{-1}b - u^{-2}a^{-1}$,

where we are abbreviating the elements $a_1, b_1 \in \mathcal{H}$ as a, b respectively.

Proof. As in the proof of Theorem 61, this reduces to computing $\langle \mathcal{T}_\gamma(e_i), e'_j \rangle$ as e_i and e'_j run through the ordered bases (58).

First note that the basis elements come in three types: those entirely supported in the genus-1 subsurface containing γ (the first three), those supported partially in this subsurface and partially in the complementary genus-$(g-1)$ subsurface (the next $4g-4$) and those supported entirely outside of the genus-1 subsurface (the rest). The Dehn twist T_γ does not mix these two complementary subsurfaces, so M_γ is a block matrix with respect to this partition.

The top-left 3×3 matrix involves only the basis elements $w(\alpha_1)$, $w(\beta_1)$, $v(\alpha_1, \beta_1)$ and their duals, and so the calculation of this submatrix is identical to the calculation in genus 1, which is given by the matrix in Figure 8.

The bottom-right submatrix involves only basis elements supported outside of the genus-1 subsurface containing γ, so the effect of $\mathcal{T}_\gamma$ is the identity on these elements.

It remains to calculate the middle $(4g-4) \times (4g-4)$ submatrix, which records the effect of $\mathcal{T}_\gamma$ on $v(\alpha_1, \epsilon)$ and $v(\beta_1, \epsilon)$ for $\epsilon \in \{\alpha_2, \ldots, \beta_g\}$. Since $\epsilon \cap \gamma = \varnothing$, we must have

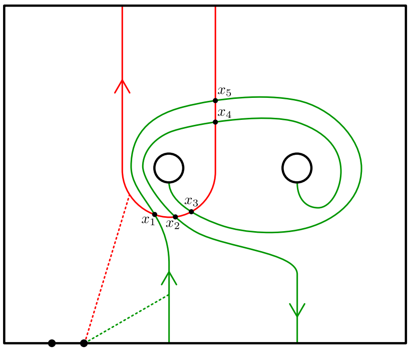
$$\mathcal{T}_\gamma(v(\alpha_1, \epsilon)) = p_\epsilon . v(\alpha_1, \epsilon) + q_\epsilon . v(\beta_1, \epsilon)$$
$$\mathcal{T}_\gamma(v(\beta_1, \epsilon)) = r_\epsilon . v(\alpha_1, \epsilon) + s_\epsilon . v(\beta_1, \epsilon)$$

for some $p_\epsilon, q_\epsilon, r_\epsilon, s_\epsilon \in \mathbb{Z}[\mathcal{H}]$. Precisely, we have

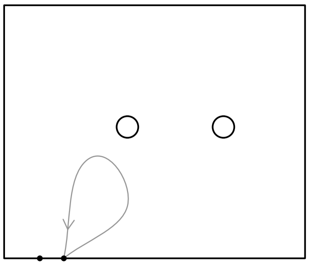
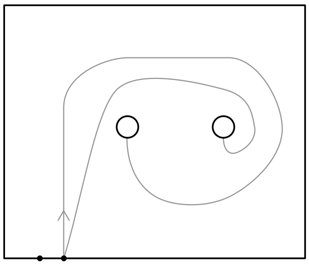
$$p_\epsilon = \big\langle v(T_\gamma(\alpha_1), \epsilon), v(\alpha'_1, \epsilon') \big\rangle \qquad q_\epsilon = \big\langle v(T_\gamma(\alpha_1), \epsilon), v(\beta'_1, \epsilon') \big\rangle$$
$$r_\epsilon = \big\langle v(T_\gamma(\beta_1), \epsilon), v(\alpha'_1, \epsilon') \big\rangle \qquad s_\epsilon = \big\langle v(T_\gamma(\beta_1), \epsilon), v(\beta'_1, \epsilon') \big\rangle,$$

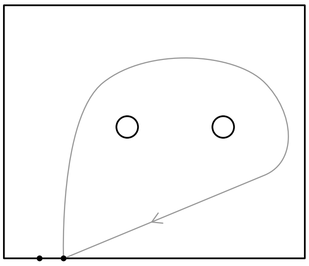
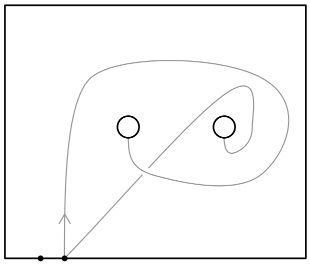
where ϵ' denotes the dual of ϵ, and we have again used the fact that $\epsilon \cap \gamma = \varnothing$ to rewrite $\mathcal{T}_\gamma(v(\alpha_1, \epsilon)) = v(T_\gamma(\alpha_1), T_\gamma(\epsilon)) = v(T_\gamma(\alpha_1), \epsilon)$ and similarly for $\mathcal{T}_\gamma(v(\alpha_1, \epsilon))$. From these formulas and (55) it is clear that $p_\epsilon, q_\epsilon, r_\epsilon, s_\epsilon$ do not in fact depend on ϵ. Indeed, when computing these values of the non-degenerate pairing, we may ignore one of the two configuration points (the one that starts on the left in the base configuration and which travels via the arcs ϵ and ϵ'), since it contributes neither to the signs nor to the loops ℓ_p in the formula (55). We will compute $s_\epsilon = s$, leaving the computation of the other three polynomials as exercises for the reader. In the following computations, as mentioned above, we ignore one of the two configuration points, since it does not contribute anything non-trivial to the formula (55).

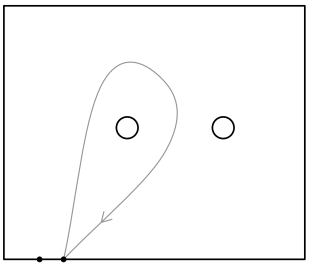
$$s = \langle v(T_\gamma(\beta_1), \epsilon), v(\beta_1', \epsilon') \rangle$$

$=$ 

(5 intersection points: $x_1, \ldots, x_5$)

$= \phi\Bigg($  $\Bigg) - \phi\Bigg($  $\Bigg)$

$+ \phi\Bigg($  $\Bigg) + \phi\Bigg($  $\Bigg)$

$- \phi\Bigg($  $\Bigg)$

$$\begin{aligned} &= \phi(\ ) - \phi(\sigma^{-1}b^{-1}aba^{-1}bab^{-1}a^{-1}b\sigma) + \phi(\sigma^{-1}ab^{-1}a^{-1}b\sigma) \\ &\quad + \phi(\sigma^{-1}a^{-1}bab^{-1}a^{-1}b\sigma) - \phi(\sigma^{-1}b^{-1}a^{-1}b\sigma) \\ &= 1 - b + u^{-2} + u^{-2}a^{-1}b - u^{-2}a^{-1}. \end{aligned}$$ ∎

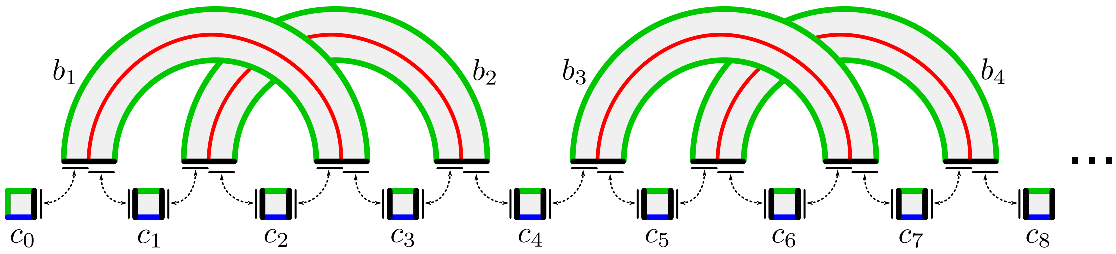


Figure 12: A model for Σ

Appendix A: a deformation retraction through Lipschitz embeddings

Here we will prove Lemma 12. We have a model for (Σ, Γ) by gluing $2g$ bands $b_j = [-1,1] \times [-l,l]$, $1 \leq j \leq 2g$ and $4g+1$ squares $c_\nu = [0,1] \times [0,1]$, $0 \leq j \leq 4g$ according to the identifications depicted in Figure 12. We obtain a deformation retraction h which is defined on each band by the formula $h_t(u,v) = ((1-t)u, v)$ and on each square by $h_t(u,v) = (u,(1-t)v)$. It remains to show that for an appropriate metric d the map h_t, $0 \leq t < 1$, is a 1-Lipschitz embedding. On each band and square we use the standard Euclidean metric. Then for points $x, y \in \Sigma$, the distance $d(x,y)$ is defined as the shortest length of a path from x to y. It is convenient to assume that l is big enough so that no shortest path can go across a handle. Then d is a metric which is flat outside $4g$ boundary points where the curvature is concentrated. Then we have that h_t, $0 \leq t < 1$, is a 1-Lipshitz embedding in each band or square from which we deduce that h_t, $0 \leq t < 1$, is globally a 1-Lipschitz embedding.

Appendix B: automorphisms of the Heisenberg group

In this appendix we prove Lemma 15. Denote $H^* = \mathrm{Hom}(H, \mathbb{Z})$. There is an obvious action of $\mathrm{Aut}(H) = GL(H)$ (and hence of $Sp(H) \subseteq \mathrm{Aut}(H)$) on H^* by pre-composition, and we consider the semi-direct product $H^* \rtimes Sp(H)$ with respect to this action. There is a well-defined homomorphism

$$H^* \rtimes Sp(H) \longrightarrow \mathrm{Aut}^+(\mathcal{H}) \tag{60}$$

given by sending $(f \colon H \to \mathbb{Z}, g \colon H \to H)$ to the automorphism of $\mathcal{H} = \mathbb{Z} \times H$ that sends $(1,0)$ to itself and $(0,x)$ to $(f(x), g(x))$ for each $x \in H$. This fits into a commutative diagram of the form

$$
\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
H^* & \longrightarrow & \mathrm{Aut}^H(\mathcal{H}) \\
\downarrow & & \downarrow \\
H^* \rtimes Sp(H) & \xrightarrow{(60)} & \mathrm{Aut}^+(\mathcal{H}) \\
\curvearrowleft\ \downarrow & & \downarrow \\
Sp(H) & \xrightarrow{\mathrm{incl}} & \mathrm{Aut}(H) \\
\downarrow & & \\
1 & &
\end{array}
$$

whose columns are exact, and where $\mathrm{Aut}^H(\mathcal{H})$ denotes the automorphisms of $\mathcal{H}$ that send $u = (1, 0)$ to itself and induce the identity on $H = \mathcal{H}/\mathcal{Z}(\mathcal{H})$. It is easy to verify by hand that (1) the top horizontal map $H^* \to \mathrm{Aut}^H(\mathcal{H})$ is bijective and (2) the image of the vertical map $\mathrm{Aut}^+(\mathcal{H}) \to \mathrm{Aut}(H)$ is contained in $Sp(H)$. These two facts imply that, if we replace the bottom-right group $\mathrm{Aut}(H)$ with $Sp(H)$, the diagram above becomes a map of short exact sequences

$$
\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
H^* & \xrightarrow{\cong} & \mathrm{Aut}^H(\mathcal{H}) \\
\downarrow & & \downarrow \\
H^* \rtimes Sp(H) & \xrightarrow{(60)} & \mathrm{Aut}^+(\mathcal{H}) \\
\curvearrowleft\ \downarrow & & \downarrow \\
Sp(H) & \xrightarrow{\mathrm{id}} & Sp(H) \\
\downarrow & & \downarrow \\
1 & & 1
\end{array}
$$

and so the five-lemma implies that (60) is an isomorphism. We record this as:

Lemma 67. *The homomorphism* (60) *is an isomorphism.*

We note that the inverse of (60) may be described as follows. By commutativity of the bottom square of the diagram above, the homomorphism

$$\mathrm{pr}_2 \circ (60)^{-1} \colon \mathrm{Aut}^+(\mathcal{H}) \longrightarrow H^* \rtimes Sp(H) \twoheadrightarrow Sp(H)$$

coincides with the natural projection $\mathrm{Aut}^+(\mathcal{H}) \to Sp(H)$. The function (crossed homomorphism)

$$\mathrm{pr}_1 \circ (60)^{-1} \colon \mathrm{Aut}^+(\mathcal{H}) \longrightarrow H^* \rtimes Sp(H) \twoheadrightarrow H^*$$

is given by sending an automorphism φ to $\mathrm{pr}_1(\varphi(0,-))\colon H \hookrightarrow \mathcal{H} \to \mathcal{H} \twoheadrightarrow \mathbb{Z}$. Putting these together, we recover precisely the description of the homomorphism $\mathrm{Aut}^+(\mathcal{H}) \to H^* \rtimes Sp(H)$ given just before the statement of Lemma 15. Thus Lemma 67 is equivalent to Lemma 15 and we have (16) = (60)$^{-1}$.

Remark 68. The arguments in this appendix apply more generally to any 2-nilpotent group. Suppose that G is a 2-nilpotent group, specifically the central extension

$$1 \to Z \longrightarrow G \longrightarrow H \to 1$$

of an abelian group H associated to a given 2-cocycle $\omega\colon H \times H \to Z$. Then there is a natural isomorphism

$$\mathrm{Aut}^Z(G) \cong \mathrm{Hom}(H, Z) \rtimes \mathrm{Aut}_\omega(H),$$

where $\mathrm{Aut}^Z(G) \subseteq \mathrm{Aut}(G)$ is the group of automorphisms of G that restrict to the identity on Z, and $\mathrm{Aut}_\omega(H) \subseteq \mathrm{Aut}(H)$ is the group of automorphisms of H that preserve the 2-cocycle ω.

In particular, we may consider the continuous Heisenberg group $\mathcal{H}_\mathbb{R}$ (used in §6): this is the central extension of $H_\mathbb{R} := H_1(\Sigma;\mathbb{R}) \cong \mathbb{R}^{2g}$ by $\mathbb{R}$ corresponding to the intersection form ω on $H_\mathbb{R}$. By the discussion above, we have natural isomorphisms

$$\mathrm{Aut}^+(\mathcal{H}) \cong \mathrm{Hom}(H, \mathbb{Z}) \rtimes Sp(H)$$
$$\mathrm{Aut}^+(\mathcal{H}_\mathbb{R}) \cong \mathrm{Hom}(H_\mathbb{R}, \mathbb{R}) \rtimes Sp(H_\mathbb{R}),$$

where, as in the discrete case, $\mathrm{Aut}^+(\mathcal{H}_\mathbb{R})$ denotes the subgroup of $\mathrm{Aut}(\mathcal{H}_\mathbb{R})$ of automorphisms that act by the identity on the central copy of $\mathbb{R}$. There are natural inclusions $Sp(H) \hookrightarrow Sp(H_\mathbb{R})$ and $\mathrm{Hom}(H,\mathbb{Z}) \hookrightarrow \mathrm{Hom}(H_\mathbb{R},\mathbb{R})$ given by tensoring $-\otimes_\mathbb{Z} \mathbb{R}$ (they are injective since $\mathbb{R}$ is torsion-free, hence flat over $\mathbb{Z}$). Together with the natural isomorphisms above, they induce a natural inclusion

$$\mathrm{Aut}^+(\mathcal{H}) \longrightarrow \mathrm{Aut}^+(\mathcal{H}_\mathbb{R}), \quad \varphi \mapsto \varphi_\mathbb{R} \tag{61}$$

having the property that, for any $\varphi \in \mathrm{Aut}^+(\mathcal{H})$, the automorphism $\varphi_\mathbb{R}\colon \mathcal{H}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$ sends $\mathcal{H} \subset \mathcal{H}_\mathbb{R}$ onto itself and restricts to $\varphi\colon \mathcal{H} \to \mathcal{H}$.

Appendix C: signs in the intersection pairing formula

Here we explain the signs appearing in the formula (55) for the intersection pairing on the homology of 2-point configuration spaces, including the extra global -1 sign that was suppressed in (55) (see the comment in the paragraph below the formula).

We take the viewpoint that an orientation o of a d-dimensional smooth manifold M is given by a consistent choice of vector $o(p) \in \Lambda^d T_p M$ for all $p \in M$. We either choose a metric on the bundle $\Lambda^d TM$ and require $o(p)$ to be a unit vector with respect to this metric, or we consider $o(p)$ up to rescaling by positive real numbers.

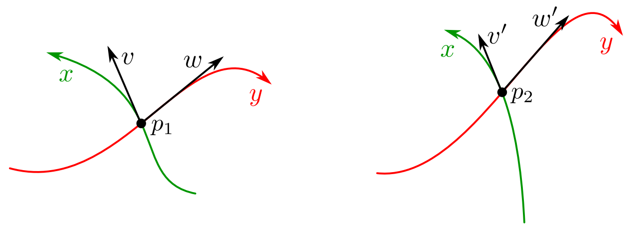


Figure 13: Choices of tangent vectors from the computation of the sign of the intersection of x and y at $p = \{p_1, p_2\} \in \mathcal{C}_2(\Sigma)$.

Let us fix an orientation o_Σ for the surface Σ. This determines an orientation $o_{\mathcal{C}_2(\Sigma)}$ of the configuration space $\mathcal{C}_2(\Sigma)$ by setting

$$o_{\mathcal{C}_2(\Sigma)}(\{p_1, p_2\}) = o_\Sigma(p_1) \wedge o_\Sigma(p_2).$$

Recall that we have 2-dimensional submanifolds x and y of $\mathcal{C}_2(\Sigma)$ that intersect transversely, and let $p = \{p_1, p_2\}$ be a point of $x \cap y$. Let v, w be the tangent vectors at p_1 and let v', w' be the tangent vectors at p_2 illustrated in Figure 13. We have

$$v \wedge w = \mathrm{sgn}(p_1).o_\Sigma(p_1)$$
$$v' \wedge w' = \mathrm{sgn}(p_2).o_\Sigma(p_2),$$

where $\mathrm{sgn}(p_i)$ is the sign of the intersection of the arcs in Σ underlying x and y at p_i. Similarly, we have

$$o_x(p) \wedge o_y(p) = \mathrm{sgn}(p).o_{\mathcal{C}_2(\Sigma)}(p),$$

where $\mathrm{sgn}(p)$ is the sign that we are trying to compute: the sign of the intersection of x and y in the configuration space. The orientations of x and y depend on the tethers t_x, t_y that have been chosen. Precisely, we have

$$o_x(p) = \begin{Bmatrix} v \wedge v' & (*) \\ v' \wedge v & (\dagger) \end{Bmatrix} \qquad\qquad o_y(p) = \begin{Bmatrix} w \wedge w' & (*) \\ w' \wedge w & (\dagger) \end{Bmatrix},$$

where the possibilities $((*), (*))$ or $((\dagger), (\dagger))$ occur if $\mathrm{sgn}(\ell_p) = +1$ and the possibilities $((*), (\dagger))$ or $((\dagger), (*))$ occur if $\mathrm{sgn}(\ell_p) = -1$. We therefore have

$$o_x(p) \wedge o_y(p) = \mathrm{sgn}(\ell_p).(v \wedge v') \wedge (w \wedge w').$$

Putting this together with the formulas above, we obtain

$$\begin{aligned}(v \wedge w) \wedge (v' \wedge w') &= \mathrm{sgn}(p_1).\mathrm{sgn}(p_2).o_\Sigma(p_1) \wedge o_\Sigma(p_2) \\ &= \mathrm{sgn}(p_1).\mathrm{sgn}(p_2).o_{\mathcal{C}_2(\Sigma)}(p) \\ &= \mathrm{sgn}(p_1).\mathrm{sgn}(p_2).\mathrm{sgn}(p).o_x(p) \wedge o_y(p) \\ &= \mathrm{sgn}(p_1).\mathrm{sgn}(p_2).\mathrm{sgn}(p).\mathrm{sgn}(\ell_p).(v \wedge v') \wedge (w \wedge w') \\ &= -\mathrm{sgn}(p_1).\mathrm{sgn}(p_2).\mathrm{sgn}(p).\mathrm{sgn}(\ell_p).(v \wedge w) \wedge (v' \wedge w'),\end{aligned}$$

and hence we have

$$\mathrm{sgn}(p) = -\mathrm{sgn}(p_1).\mathrm{sgn}(p_2).\mathrm{sgn}(\ell_p).$$

Appendix D: Sage computations

Here we give the worksheet of the Sage computations used in the calculation of the matrix M_∂ displayed in Figure 8 (cf. Remark 64 on page 46).

In [1]:
```
load("HeisLatex_.sage") #available on demand
```

In [2]:
```
# R is the center of Heisenberg group ring
R.<u>= LaurentPolynomialRing(ZZ,1)
```

In [3]:
```
# H is Heisenberg group ring
H = Heis(base=R, category=Rings())
```

In [32]:
```
a=H(dict({(1,0):1}))   #generator (0,a)
b=H(dict({(0,1):1}))
am=H(dict({(-1,0):1})) #inverse generators
bm=H(dict({(0,-1):1}))
```

In [33]:
```
a*b-u^2*b*a #check relation
```

Out[33]: 0

In [34]:
```
# a->a , b -> a^-1b (T_a action on H)
def Ha(h:HeisEl):
    d0=h.d
    h1=H()
    for k in d0:
        i=k[0]
        j=k[1]
        h1+= H({(i-j,j):d0[k]*u^(j*(j-1))})
    return h1
def MHa(M): # same on matrices
    M1=matrix(H,3)
    for i in range(3):
        for j in range(3):
            M1[i,j]=Ha(M[i,j])
    return M1
```

In [35]:
```
def Hb(h:HeisEl): #T_b action on H
    d0=h.d
    h1=H()
    for k in d0:
        i=k[0]
        j=k[1]
        h1+= H({(i,i+j):d0[k]*u^(-i*(i-1))})
    return h1
def MHb(M): # same on matrices
    M1=matrix(H,3)
    for i in range(3):
        for j in range(3):
            M1[i,j]=Hb(M[i,j])
    return M1
```

In [36]:
```
def Hab(h):       #other actions
    return Ha(Hb(h))
def Hba(h):
    return Hb(Ha(h))
def Haba(h):
    return Ha(Hba(h))
def Hbab(h):
    return Hb(Hab(h))
def Hs(h):
    return(Haba(Haba(h)))
```

In [37]:
```
def MHab(M):    #same on matrices
    return MHa(MHb(M))
def MHba(M):
    return MHb(MHa(M))
def MHaba(M):
    return MHa(MHba(M))
def MHbab(M):
    return MHb(MHab(M))
def MHs(M):
    return(MHaba(MHaba(M)))
```

In [38]:
```
Ma=matrix([[H(1),u^2*am^(2)*b^2,(H(u^(-1))-H(1))*am*b],[H(0),H(1),H(0)],[H(0),H(-1)*am*b,H(1)]])
```

In [39]:
```
%display latex
Ma #Ta action
```

Out[39]:
$$\begin{pmatrix} 1 & u^2a^{-2}b^2 & (-1+u^{-1})a^{-1}b^1 \\ 0 & 1 & 0 \\ 0 & -a^{-1}b^1 & 1 \end{pmatrix}$$

In [40]:
```
Mb=matrix([[H(u^(-2))*b^2,H(0),H(0)],[H(-u^(-1)),H(1),H(1-u^(-1))],[H(-u^(-1))*b,H(0),b]])
```

In [41]:
```
Mb #Tb action
```

Out[41]:
$$\begin{pmatrix} u^{-2}b^2 & 0 & 0 \\ -u^{-1} & 1 & 1-u^{-1} \\ -u^{-1}b^1 & 0 & b^1 \end{pmatrix}$$

In [42]:
```
MHa(Mb) # Ta shifted action of Tb
```

Out[42]:
$$\begin{pmatrix} a^{-2}b^2 & 0 & 0 \\ -u^{-1} & 1 & 1-u^{-1} \\ -u^{-1}a^{-1}b^1 & 0 & a^{-1}b^1 \end{pmatrix}$$

In [43]:
```
MHb(Ma) #Tb shifted action of Ta
```

Out[43]:
$$\begin{pmatrix} 1 & u^{-4}a^{-2} & (-u^{-2}+u^{-3})a^{-1} \\ 0 & 1 & 0 \\ 0 & -u^{-2}a^{-1} & 1 \end{pmatrix}$$

In [44]:
```
MHab(Ma) #TaTb shifted action of Ta
```

Out[44]:
$$\begin{pmatrix} 1 & u^{-4}a^{-2} & (-u^{-2}+u^{-3})a^{-1} \\ 0 & 1 & 0 \\ 0 & -u^{-2}a^{-1} & 1 \end{pmatrix}$$

In [45]:
```
MHba(Mb) #TbTa shifted action of Tb
```
Out[45]:
$$\begin{pmatrix} u^{-6}a^{-2} & 0 & 0 \\ -u^{-1} & 1 & 1-u^{-1} \\ -u^{-3}a^{-1} & 0 & u^{-2}a^{-1} \end{pmatrix}$$

In [46]:
```
X=Ma*MHa(Mb)*MHab(Ma) #action of TaTbTa
X
```
Out[46]:
$$\begin{pmatrix} 0 & u^{2}a^{-2}b^{2} & 0 \\ -u^{-1} & -u^{-5}a^{-2}+1+(-u^{-2}+u^{-3})a^{-1} & (u^{-3}-u^{-4})a^{-1}+1-u^{-1} \\ 0 & -a^{-1}b^{1}-u^{-1}a^{-2}b^{1} & u^{-1}a^{-1}b^{1} \end{pmatrix}$$

In [47]:
```
Y=Mb*MHb(Ma)*MHba(Mb) #action of TbTaTb
Y
```
Out[47]:
$$\begin{pmatrix} 0 & u^{2}a^{-2}b^{2} & 0 \\ -u^{-1} & -u^{-5}a^{-2}+1+(-u^{-2}+u^{-3})a^{-1} & 1-u^{-1}+(u^{-3}-u^{-4})a^{-1} \\ 0 & -u^{-1}a^{-2}b^{1}-a^{-1}b^{1} & u^{-1}a^{-1}b^{1} \end{pmatrix}$$

In [48]:
```
X-Y #check braid relation TaTbTa=TbTaTb
```
Out[48]:
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In [49]:
```
Z=X*MHaba(X) #action of (TaTbTa)^2
```

In [50]:
```
Z[:,0] # first colomn
```
Out[50]:
$$\begin{pmatrix} -ua^{-2}b^{2} \\ u^{-6}a^{-2}+-u^{-1}+(u^{-3}-u^{-4})a^{-1} \\ u^{-1}a^{-1}b^{1}+u^{-2}a^{-2}b^{1} \end{pmatrix}$$

In [51]:
```
Z[:,1]
```
Out[51]:
$$\begin{pmatrix} -u^{-3}a^{-2}+u^{2}a^{-2}b^{2}+(-1+u^{-1})a^{-2}b^{1} \\ -u^{-5}b^{-2}+-u^{-5}a^{-2}+1+(-u^{-2}+u^{-3})a^{-1}+(-u^{-2}+u^{-3})b^{-1}+(-u^{-5}+u^{-6})a^{-1}b^{-1} \\ u^{-5}a^{-1}b^{-1}-a^{-1}b^{1}+-u^{-1}a^{-2}b^{1}+(u^{-2}-u^{-3})a^{-1}+-u^{-4}a^{-2} \end{pmatrix}$$

In [52]:
```
Z[:,2]
```
Out[52]:
$$\begin{pmatrix} (u^{-1}-u^{-2})a^{-2}b^{1}+(u^{2}-u)a^{-2}b^{2} \\ (u^{-3}-u^{-4})b^{-1}+(u^{-6}-u^{-7})a^{-1}b^{-1}+(-u^{-5}+u^{-6})a^{-2}+1-u^{-1}+(-u^{-2}+2u^{-3}-u^{-4})a^{-1} \\ (-u^{-3}+u^{-4})a^{-1}+u^{-5}a^{-2}+(-1+u^{-1})a^{-1}b^{1}+(-u^{-1}+u^{-2})a^{-2}b^{1} \end{pmatrix}$$

In [53]:
```
ZZ=Z*MHs(Z) #action of Tc=(TaTbTa)^4
```

In [54]:
```
ZZ[:,0]
```
Out[54]:
$$\begin{pmatrix} u^{-8}b^{2}+u^{-4}a^{-2}+-ua^{-2}b^{2}+(u^{-1}-u^{-2})a^{-2}b^{1}+(u^{-3}-u^{-4})a^{-1}b^{2}+(u^{-4}-u^{-5})a^{-1}b^{1} \\ -u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^{2}+(u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{1}+u^{-6}a^{-2}+(u^{-3}-u^{-4}-u^{-6}+u^{-7})a^{-1} \\ -u^{-6}a^{1}b^{1}+(-u^{-3}+u^{-4}-u^{-7})b^{1}+-u^{-4}+(u^{-1}-u^{-4}+u^{-5})a^{-1}b^{1}+u^{-2}a^{-2}b^{1}+(-u^{-3}+u^{-6})a^{-1}+u^{-5}a^{-2} \end{pmatrix}$$

In [55]:
```
ZZ[:,1]
```
Out[55]:
$$\begin{pmatrix} (u^{2}+1-2u^{-1}+u^{-2}+u^{-4})a^{-2}b^{2}+-ua^{-2}b^{4}+(-u^{2}+u+u^{-1}-u^{-2})a^{-2}b^{3}+-u^{-3}a^{-2}+(-1+u^{-1}+u^{-3}-u^{-4})a^{-2}b^{1} \\ 1+u^{-2}-u^{-3}+u^{-6}+u^{-6}a^{-2}b^{2}+-u^{-1}b^{2}+(u^{-3}-u^{-4})a^{-1}b^{2}+(-1+u^{-1}+u^{-3}-u^{-4})b^{1} \\ +(u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})a^{-1}b^{1}+-u^{-5}a^{-2}+(-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}+(u^{-5}-u^{-6})a^{-2}b^{1} \\ (-1-u^{-2}+2u^{-3}-u^{-6})a^{-1}b^{1}+u^{-1}a^{-1}b^{3}+u^{-2}a^{-2}b^{3}+(1-u^{-1}-u^{-3}+u^{-4})a^{-1}b^{2}+(u^{-1}-u^{-2}+u^{-5})a^{-2}b^{2} \\ +(-u^{-1}+u^{-4}-u^{-5})a^{-2}b^{1}+(u^{-2}-u^{-5})a^{-1}+-u^{-4}a^{-2} \end{pmatrix}$$

In [56]:
```
ZZ[:,2]
```
Out[56]:
$$\begin{pmatrix} (-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-2}b^{1}+(u-1)a^{-2}b^{3}+(u^{2}-u-u^{-1}+2u^{-2}-u^{-3})a^{-2}b^{2}+(-u^{-3}+u^{-4})a^{-1}b^{1} \\ +(u^{-4}-u^{-5})a^{-1}b^{3}+(-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}b^{2}+(-u^{-3}+u^{-4})a^{-2} \\ (-u^{-6}+u^{-7})a^{-2}b^{1}+(u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b^{1}+(-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})a^{-1}b^{1}+1-u^{-1}+u^{-2}-3u^{-3} \\ +2u^{-4}+u^{-6}-u^{-7}+(-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a^{-1}+(u^{-2}-u^{-3})a^{1}b^{1}+(-1+u^{-1}+u^{-3}-u^{-4})a^{1} \\ +(-u^{-5}+u^{-6})a^{-2} \\ u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a^{-1}+(-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-1}b^{2}+(-u^{-2}+u^{-3})a^{-2}b^{2} \\ +(-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})a^{-1}b^{1}+(-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-2}b^{1}+(-u^{-4}+u^{-5})b^{2}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})b^{1} \\ +(-u^{-4}+u^{-5})a^{-2} \end{pmatrix}$$

In [57]:
```
ZZ*Ma-Ma*MHa(ZZ) # check that Tc is central
```
Out[57]:
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In [58]:
```
ZZ*Mb-Mb*MHb(ZZ)
```
Out[58]:
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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