When the lower central series stops

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Abstract

Our aim here is to showcase several techniques for studying the lower central series of a group and, in particular, for determining whether or not it stops. We apply these techniques to various groups related to braid groups, in particular Artin groups, surface braid groups, groups of welded and virtual braids and *partitioned* versions of all of these groups.

Introduction

One of the most basic objects one needs to understand when studying the structure of a group G is its lower central series $G = \Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \cdots$. Its behaviour varies greatly from one group to another. For instance, if G is perfect, its lower central series is completely trivial. Conversely, if G is nilpotent, or residually nilpotent (e.g. if G is a free group), $\Gamma_*(G)$ contains deep information about the structure of G. The lower central series is also deeply connected to the structure of the group ring of G. In particular, Quillen [Qui68] proved that if we consider the filtration of the group ring $\mathbb{Q}G$ by the powers of its augmentation ideal, then the associated graded algebra is isomorphic to the universal enveloping algebra of the Lie algebra $\mathcal{L}(G) \otimes \mathbb{Q}$, where $\mathcal{L}(G)$ is the graded Lie ring obtained from $\Gamma_*(G)$.

The amount of information one can hope to extract from the study of a lower central series depends in the first place on whether or not it *stops* in the following sense:

Definition 0.1 The lower central series $\Gamma_*(G)$ of a group G is said to *stop* if there exists an integer $i \geqslant 1$ such that $\Gamma_i(G) = \Gamma_{i+1}(G)$. We say that it *stops at Γ_i* if i is the smallest integer for which this holds. Otherwise, we say the lower central series *does not stop* or else that it *stops at ∞*.

It follows from the definition of the lower central series that if $\Gamma_i(G) = \Gamma_{i+1}(G)$ for some $i \geqslant 1$, then $\Gamma_k(G) = \Gamma_{k+1}(G)$ for all $k \geqslant i$, whence our choice of terminology.

Partitioned braid groups. In this paper, we study the lower central series of the following families of groups, and their *partitioned* versions, in the sense described below:

- The Artin braid group $\mathbf{B}_n$,
- The virtual braid group $\mathbf{vB}_n$,
- The welded braid group $\mathbf{wB}_n$,
- The extended welded braid group $\widetilde{\mathbf{w}}\mathbf{B}_n$,
- The group $\mathbf{B}_n(S)$ of braids on any surface S.

Let G_n denote one of the above groups. In each case, there is a notion of the underlying permutation of an element of G_n, corresponding to a canonical surjection $\pi \colon G_n \twoheadrightarrow \mathfrak{S}_n$ to the symmetric group, from which we can define *partitioned* versions of G_n. Let us first fix our conventions concerning partitions of integers:

Definition 0.2 Let $n \geqslant 1$ be an integer. A *partition of n* is an *l*-tuple $\lambda = (n_1, \ldots, n_l)$ of integers $n_i \geqslant 1$, for some $l \geqslant 1$ called the *length* of λ, such that n is the sum of the n_i. Given such a λ, for $j \leqslant l$, let us define $t_j := \sum_{i \leqslant j} n_i$, including $t_0 = 0$. Then the set $b_j(\lambda) := \{t_{j-1} + 1, \ldots, t_j\}$ is referred to as the *j-th block* of λ, and n_i is called the *size* of the i-th block.

For $\lambda = (n_1, \ldots, n_l)$ a partition of n, we consider the preimage

$$G_\lambda := \pi^{-1}(\mathfrak{S}_\lambda) = \pi^{-1}\left(\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}\right),$$

which is called the *λ-partitioned* version of G_n. There are two extremal situations: the trivial partition $\lambda = (n)$ simply gives the group G_n, whereas the discrete partition $\lambda = (1, 1, \ldots, 1)$ corresponds to the subgroup of *pure* braids in G_n.

For all the families of groups described above, the lower central series of G_n is quite trivial (in fact, it stops at Γ_2 when n is at least 3 or 4), whereas the lower central series of the subgroup of pure braids is a very complex object (in particular, it does not stop, when n is at least 2 or 3). We can thus expect the partitioned braid groups G_λ to display a range of intermediate behaviours when λ varies, and this is indeed what we observe.

Methods. A fundamental tool in the study of lower central series is the graded Lie ring structure on the associated graded $\mathcal{L}(G) := \bigoplus_{i \geqslant 1} \Gamma_i(G)/\Gamma_{i+1}(G)$. Namely, this is a graded abelian group endowed with a Lie bracket induced by commutators in G. It is always generated, as a Lie algebra over $\mathbb{Z}$, by its degree one piece, which is the abelianisation $G^{\mathrm{ab}} = G/\Gamma_2(G)$. This allows one to use *disjoint support arguments* to show that the lower central series stops, when it does. Precisely, if one can show that pairs of generators of G^{ab} have commuting representatives in G, then, by definition of the Lie bracket, they commute in $\mathcal{L}(G)$. In this case, $\mathcal{L}(G)$ is abelian, and it is generated by G^{ab}, which means that it is reduced to G^{ab}. In turn, that means that $\Gamma_i(G) = \Gamma_{i+1}(G)$ whenever $i \geqslant 2$. This kind of argument is used throughout the paper. In particular, one can apply it readily to each of the G_n above to show that its lower central series stops at Γ_2 whenever n is at least 3 or 4 (depending on the amount of space needed to have representatives with disjoint support for pairs of generators of G^{ab}).

One other main line of argumentation is given by *looking for quotients whose lower central series is well-understood.* Namely, if we can find a quotient of G whose lower central series does not stop, then neither does the lower central series of G; see Lemma 1.1. Typically, we look for a quotient which is a semi-direct product of an abelian group with $\mathbb{Z}$ or $\mathbb{Z}/2$, a free product of abelian groups or a wreath product of an abelian group with some $\mathfrak{S}_\lambda$, whose lower central series can be computed completely; see §B.

Finally, a very important tool in our analysis is the study of the *quotient by the residue.* Precisely, if we denote by $\Gamma_\infty(G)$ (abbreviated Γ_∞ when the context is clear) the intersection of the $\Gamma_i(G)$, the lower central series of G is "the same" as the lower central series of G/Γ_∞: each $\Gamma_i(G)$ is the preimage of $\Gamma_i(G/\Gamma_\infty)$ by the canonical projection, and this projection induces an isomorphism between $\mathcal{L}(G)$ and $\mathcal{L}(G/\Gamma_\infty)$. In particular, one of $\Gamma_*(G)$ and $\Gamma_*(G/\Gamma_\infty)$ stops if and only if the other does, which happens exactly when G/Γ_∞ is nilpotent. Considering G/Γ_∞ instead of G can lead to very important simplifications. Let us illustrate this by an example, variations of which are used throughout the paper. We know that $\Gamma_\infty = \Gamma_2$ for $\mathbf{B}_n$, and that Γ_2 contains the elements $\sigma_i \sigma_j^{-1}$ of $\mathbf{B}_n$. Thus whenever we have a morphism $\mathbf{B}_n \to G$, the subgroup $\Gamma_\infty(G)$ must contain the image of $\Gamma_\infty(\mathbf{B}_n)$, which contains the image of the $\sigma_i \sigma_j^{-1}$, so all the σ_i have the same image in G/Γ_∞.

Results. Does a given lower central series stop? We give a complete answer to this question for all of the families of groups listed above, with the single exception of $\mathbf{B}_{2,m}(\mathbb{P}^2)$ with $m \geqslant 3$ (see Conjecture 6.71). In addition to these families of groups, we also give a complete answer for what we call the *tripartite* welded braid groups $\mathbf{wB}(n_P, n_{S_+}, n_S)$, which are the fundamental groups of the configuration spaces of n_P points, n_{S_+} oriented circles and n_S unoriented circles in 3-space, where all circles are unlinked and unknotted. This is a generalisation of both $\mathbf{wB}_n = \mathbf{wB}(0, n, 0)$

and $\widetilde{\mathbf{w}}\mathbf{B}_n = \mathbf{w}\mathbf{B}(0, 0, n)$. We define partitioned versions $\mathbf{w}\mathbf{B}(\lambda_P, \lambda_{S_+}, \lambda_S)$ of these groups in the obvious way; see Definition 5.5.

We summarise our results in the three tables below:

- In the first one are gathered the *stable* cases, which are those where the blocks of the partitions are large enough for the *disjoint support argument* described above to be applied readily.
- In the second one are gathered the cases where there are blocks in the partitions which are too small for the disjoint support argument to be applied readily, but not too many of them, so that the lower central series still stops.
- In the third one are gathered the cases where the lower central series does not stop.

Some of our results have already been obtained in the literature, with different methods. Namely, the question of whether or not the lower central central series stops has already been studied:

- by Gorin and Lin [GL69] for $\mathbf{B}_n$ and Kohno [Koh85] for the pure braid group $\mathbf{P}_n = \mathbf{B}_{1,\dots,1}$.
- by Bellingeri, Gervais and Guaschi [BGG08] for $\mathbf{B}_n(S)$ where S is a compact, connected, orientable surface with or without boundary.
- by Bellingeri and Gervais [BG16] for the pure surface braid group $\mathbf{P}_n(S)$ where S is a compact, connected, non-orientable surface with or without boundary and different from the projective plane $\mathbb{P}^2$.
- by Gonçalves and Guaschi [GG09a; GG09b] for $\mathbf{B}_n(\mathbb{S}^2)$ and $\mathbf{B}_n(\mathbb{S}^2 - \mathcal{P})$ where $\mathcal{P}$ is a finite set of points in $\mathbb{S}^2$.
- by Guaschi and de Miranda e Pereiro [GM20] for $\mathbf{B}_n(S)$ where S is a compact, connected, non-orientable surface without boundary.
- by van Buskirk [Bus66] and by Gonçalves and Guaschi [GG04a; GG11; GG07] both for the braid group on the projective plane $\mathbf{B}_n(\mathbb{P}^2)$.
- by Bardakov and Bellingeri [BB09a] for the virtual braid group $\mathbf{v}\mathbf{B}_n$.

The symbol (†) in front of a family of groups in the tables indicates that the result concerned is already partly known in the literature quoted above.

In the following tables, $\lambda = (n_1, \dots, n_l)$ denotes a partition of n, of length $l \geqslant 1$, and S denotes any connected surface (not necessarily compact or orientable, and possibly with boundary).

The stable cases					
Family of groups		Partition	Stops at Γ_k	Ref.	Lie Alg.
Classical braids $\mathbf{B}_\lambda$		$n_i \geqslant 3$ (†)	$k = 2$	3.5 (3.6)	3.4
Surface braids $\mathbf{B}_\lambda(S)$	S planar or $S = \mathbb{S}^2$	$n_i \geqslant 3$ (†)	$k = 2$	6.30	6.26
	$S \nsubseteq \mathbb{S}^2$, orientable	$n_i \geqslant 3$ (†)	$k = 3$		§6.4.3
	S non-orientable	$l = 1$, $n_1 \geqslant 3$	$k = 2$		6.11
		$l \geqslant 2$, $n_i \geqslant 3$	$k = 3$		§6.4.3
Virtual braids $\mathbf{v}\mathbf{B}_\lambda$		$n_i \geqslant 4$	$k = 2$	4.22	4.21
Welded braids $\mathbf{w}\mathbf{B}_\lambda$					
Extended welded braids $\widetilde{\mathbf{w}}\mathbf{B}_\lambda$					
Tripartite welded braids $\mathbf{w}\mathbf{B}(\lambda_P, \lambda_{S_+}, \lambda_S)$		$\begin{cases} n_{i,P} \geqslant 3 \\ n_{i,S_+}, n_{i,S} \geqslant 4 \end{cases}$	$k = 2$	5.8 (5.9)	5.7

In the following tables, μ denotes a partition which is either empty or whose blocks have size at least 3 for classical braids and surface braids, and at least 4 for virtual and welded braids, whereas ν denotes any partition (possibly empty, unless stated otherwise). Also, v_2 denotes the 2-adic valuation and ϵ is 0 or 1 (the precise value may depend on the case, and is unknown, although we conjecture that it is always 1). Finally, we recall that l denotes total length of the partition λ under consideration.

The unstable cases for which the lower central series stops

Family of groups		Partition	Stops at Γ_k	Ref.	Lie Alg.
Classical braids $\mathbf{B}_\lambda$		(2)	$k=2$	$\mathbf{B}_2 \cong \mathbb{Z}$	
		$(1,\mu),\ (1,1,\mu)$		3.5 (3.8, 3.9)	3.4
Surface braids $\mathbf{B}_\lambda(S)$	$S = \mathbb{D} - pt$	$(1,\mu)$	$k=2$	6.41	3.4
	$S = \mathbb{T}^2$	(1)	$k=2$	$\mathbf{B}_1(\mathbb{T}^2) \cong \mathbb{Z}^2$	
		$(1,\mu),\ \mu \neq \varnothing$	$k=3$	6.43	6.26 and 6.34
	$S = \mathbb{M}^2$	(1)	$k=2$	$\mathbf{B}_1(\mathbb{M}^2) \cong \mathbb{Z}$	
	$S = \mathbb{S}^2$	(2) or $(2,1)$	$k=2$	$\mathbf{B}_2(\mathbb{S}^2) \cong \mathbb{Z}/2,\ \mathbf{B}_{2,1}(\mathbb{S}^2) \cong \mathbb{Z}/4$	
		$(1,\mu),\ (1,1,\mu),\ (1,1,1,\mu)$	$k=2$	6.48	6.26
		$(2,m),\ m \geqslant 3$	$k = v_2(m)+1+\epsilon$	6.51	$-$
	$S = \mathbb{P}^2$	$(1,m),\ m \geqslant 3$	$k = v_2(m)+2+\epsilon$	6.57	$-$
		(1)	$k=2$	$\mathbf{B}_1(\mathbb{P}^2) \cong \mathbb{Z}/2$	
		$(1,1)$	$k=3$	$\mathbf{B}_{1,1}(\mathbb{P}^2) \cong Q_8$ (6.59)	
		(2)	$k=4$	$\mathbf{B}_2(\mathbb{P}^2) \cong Dic_{16}$ (6.59)	
Virtual braids $\mathbf{vB}_\lambda$		$(1,\mu)$	$k=2$	4.22 (4.26)	4.21
Welded braids $\mathbf{wB}_\lambda$				4.22 (4.24)	
Tripartite welded braids $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$		$((1,\dots,1,\mu_P),\mu_{S_+},\mu_S)$	$k=2$	5.8 (5.10)	5.7
		$((1,\dots,1,\mu_P),(1,\mu_{S_+}),\mu_S)$		5.8 (5.11, 5.12)	
		$((2,\nu_P),\varnothing,\mu_S),\ \mu_S \neq \varnothing$	$k=3$	5.8 (5.17)	5.7 and 5.17

The unstable cases for which the lower central series does not stop

Family of groups		Partition	Ref.
Classical braids $\mathbf{B}_\lambda$		$(1,1,1,\nu)$ (†)	3.5 (3.7)
		$(2,\nu),\ l \geqslant 2$	3.5 (3.11, 3.14, 3.17)
Surface braids $\mathbf{B}_\lambda(S)$	$S \notin \{\mathbb{D}, \mathbb{D} - pt, \mathbb{T}^2, \mathbb{M}^2, \mathbb{S}^2, \mathbb{P}^2\}$	$(1,\nu),\ (2,\nu)$ (†)	6.40
	$S = \mathbb{D} - pt$	$(1,1,\nu)$	6.41
		$(2,\nu)$	6.40
	$S = \mathbb{T}^2$	$(1,1,\nu)$	6.43
		$(2,\nu)$ (†)	6.43 (6.40)
	$S = \mathbb{M}^2$	$(1,\nu),\ l \geqslant 2$	6.40, 6.45, 6.46
		$(2,\nu)$	6.40
	$S = \mathbb{S}^2$	$(1,1,1,1,\nu)$	6.48 (6.49)
		$(2,\nu),\ l \geqslant 3$ or $(2,2)$	6.48 (6.50, 6.53)
	$S = \mathbb{P}^2$	$(1,\nu),\ l \geqslant 3$	6.56
		$(2,\nu),\ l \geqslant 3$ or $(2,2)$ or $(2,1)$	6.60, 6.68, 6.65
Virtual braids $\mathbf{vB}_\lambda$		$(1,1,\nu),\ (2,\nu),\ (3,\nu)$	4.22
Welded braids $\mathbf{wB}_\lambda$			
Extended welded braids $\widetilde{\mathbf{w}}\mathbf{B}_\lambda$		$[(1,\nu)$ with $\nu \neq \varnothing],\ (2,\nu),\ (3,\nu)$	
Tripartite welded braids $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$		$(\nu_P,\nu_{S_+},(2,\nu_S)),\ (\nu_P,\nu_{S_+},(3,\nu_S))$	5.8
		$(\nu_P,(2,\nu_{S_+}),\nu_S),\ (\nu_P,(3,\nu_{S_+}),\nu_S)$	
		$(\nu_P,(1,1,\nu_{S_+}),\nu_S)$	
		$(\nu_P,\nu_{S_+},(1,\nu_S)),\ l \geqslant 2$	5.8 (5.13)
		$((2,\nu_P),\nu_{S_+},\nu_S),\ \nu_{S_+} \neq \varnothing$	5.8 (5.14)

Contents

1   General recollections

In this section, we recall some classical notions and tools to study the lower central series of groups. These will be used throughout the paper.

1.1 Commutator calculus and lower central series

Let G be a group. We recall that the *lower central series* of G is the descending sequence of normal subgroups $G = \Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \cdots$ of G, also denoted by $\Gamma_*(G)$, defined by

$$\Gamma_i(G) := \begin{cases} G & \text{if } i = 1, \\ [G, \Gamma_{i-1}(G)] & \text{if } l \geqslant 2, \end{cases}$$

where $[G, \Gamma_l(G)]$ is the subgroup of G generated by all commutators $[x, y] := xyx^{-1}y^{-1}$ with x in G and y in $\Gamma_{l-1}(G)$. The subgroups $\Gamma_i(G)$ are fully invariant, and in particular normal in G. As a consequence, one can also think of the lower central series as an ascending chain of quotients $G/\Gamma_i(G)$ of G. We recall that the abelianisation G^{ab} of G is the first of these quotients, namely $G/\Gamma_2(G)$. In general, $G/\Gamma_{i+1}(G)$ is the universal i-nilpotent quotient of G.

The following easy lemma will be used very often in the sequel. In particular, we will often use its contrapositive: if a group G has a quotient whose lower central series does not stop, then the lower central series of G does not stop either.

Lemma 1.1 *Let H be a quotient of G. If $\Gamma_i(G) = \Gamma_{i+1}(G)$ for some i, then $\Gamma_i(H) = \Gamma_{i+1}(H)$.*

Proof. For all $k \geqslant 1$, it follows from the definition of the lower central series that $\Gamma_k H = \pi(\Gamma_k G)$. As a consequence, $\Gamma_{i+1} H = \Gamma_i H$ whenever $\Gamma_{i+1} G = \Gamma_i G$. $\square$

We recall that a group G is called *residually nilpotent* if its residue $\Gamma_\infty(G) := \bigcap \Gamma_i(G)$ is equal to $\{1\}$. The following result will be useful to us in our calculations of residues and quotients by residues:

Proposition 1.2 *Let G be a group, and let N be a normal subgroup of G. Suppose that for some $i \geqslant 2$, $N \cap \Gamma_i(G) = \{1\}$. Then $\Gamma_\infty(G) \to \Gamma_\infty(G/N)$ is an isomorphism. In particular, G is residually nilpotent if and only if G/N is. Moreover, we have a short exact sequence:*

$$N \hookrightarrow G/\Gamma_\infty \twoheadrightarrow (G/N)/\Gamma_\infty.$$

Proof. Let $\pi : G \twoheadrightarrow G/N$ be the corresponding projection. Since $N \cap \Gamma_\infty(G) = \{1\}$, the induced morphism $\Gamma_\infty(G) \to \Gamma_\infty(G/N)$ is injective. Let us show that it is surjective. Let $y \in \Gamma_\infty(G/N)$. Since $\Gamma_k(G/N) = \pi(\Gamma_k G)$ by definition of the lower central series, there is, for each $k \geqslant 1$, some $x_k \in \Gamma_k(G)$ such that $\pi(x_k) = y$. Then $x_k x_{k+1}^{-1} \in N \cap \Gamma_k(G)$, which implies that $x_k x_{k+1}^{-1} = 1$ whenever $k \geqslant i$. Thus the sequence (x_k) is stationary at $x := x_i$, which must be in $\Gamma_\infty(G)$ by definition of the x_k, and is sent to y by π. This proves the first part of the Proposition. Now, let us consider the commutative diagram of groups:

$$\begin{array}{ccccc} 1 & \hookrightarrow & \Gamma_\infty(G) & \xrightarrow{\cong} & \Gamma_\infty(G/N) \\ \downarrow & & \downarrow & & \downarrow \\ N & \hookrightarrow & G & \twoheadrightarrow & G/N \\ \downarrow{\scriptstyle =} & & \downarrow & & \downarrow \\ N & \dashrightarrow & G/\Gamma_\infty & \dashrightarrow & (G/N)/\Gamma_\infty. \end{array}$$

By the Nine Lemma, the bottom row must be a short exact sequence. $\square$

The following corollary provides a partial converse to Lemma 1.1:

Corollary 1.3 *Let G be a group and A be a central subgroup of G. Suppose that for some $i \geqslant 2$, the canonical map $A \to G/\Gamma_i(G)$ is injective. If the lower central series of G/A stops at Γ_k, then the lower central series of G stops at Γ_k or at Γ_{k+1}.*

Proof. The extension $A \hookrightarrow G/\Gamma_\infty \twoheadrightarrow (G/A)/\Gamma_\infty$ from Proposition 1.2 is a central one. As a consequence, if $(G/A)/\Gamma_\infty$ is k-nilpotent, then G/Γ_∞ is nilpotent of class k or $k+1$. $\qquad\square$

Remark 1.4 We will mostly apply Corollary 1.3 with $i = 2$, that is, for $A \hookrightarrow G^{\mathrm{ab}}$; see the proof of Proposition 6.51 for instance.

1.2   Lie rings of lower central series

We now recall the definition and basic properties of a key tool for studying the lower central series of a group, namely its associated Lie ring. We refer the reader to [Laz54, Chap. 1] for further details.

Note that, for all $i \geqslant 1$, $[\Gamma_i(G), \Gamma_i(G)] \subseteq [G, \Gamma_i(G)] \subseteq \Gamma_{i+1}(G) \subseteq \Gamma_i(G)$. Thus, $\Gamma_i(G)$ is a normal subgroup of G, and the quotient $\mathcal{L}_i(G) := \Gamma_i(G)/\Gamma_{i+1}(G)$ is an abelian group. Moreover, one can show that $[\Gamma_i(G), \Gamma_j(G)] \subseteq \Gamma_{i+j}(G)$ for all $i, j \geqslant 1$, which is the crucial property allowing us to define the Lie ring associated with $\Gamma_*(G)$:

Proposition 1.5 ([Laz54, Th. 2.1]) *The graded abelian group $\mathcal{L}(G) := \bigoplus_{i \geqslant 1} \mathcal{L}_i(G)$ is a Lie ring, with the Lie bracket induced by the commutator map of G.*

Convention 1.6 Let g be an element of G. If there is an integer d such that $g \in \Gamma_d(G) - \Gamma_{d+1}(G)$, it is obviously unique. We then call d the *degree* of g with respect to $\Gamma_*(G)$. The notation $\overline{g}$ denotes the class of g in some quotient $\mathcal{L}_i(G)$. If the integer i is not specified, it is assumed that $i = d$, which means that $\overline{g}$ denotes the only non-trivial class induced by g in $\mathcal{L}(G)$. If such a d does not exist (that is, if $g \in \bigcap \Gamma_i(G)$), we say that g has degree ∞ and we put $\overline{g} = 0$.

With this convention, the Lie bracket $[-,-]$ of $\mathcal{L}(G)$ is given by the collection of bilinear maps $\mathcal{L}_i(G) \times \mathcal{L}_j(G) \to \mathcal{L}_{i+j}(G)$ defined by:

$$\forall x \in \mathcal{L}_i(G), \ \forall y \in \mathcal{L}_j(G), \ [\overline{x}, \overline{y}] := \overline{[x,y]} \in \mathcal{L}_{i+j}(G).$$

The following lemma, which will be used several times in the sequel to identify G/Γ_∞ for some group G, is one illustration of the use of Lie rings in studying the lower central series:

Lemma 1.7 *Let $p \colon G \twoheadrightarrow Q$ be a surjective group morphism. If Q is a residually nilpotent group, then the following conditions are equivalent:*

- *$\mathcal{L}(p) \colon \mathcal{L}(G) \twoheadrightarrow \mathcal{L}(Q)$ is an isomorphism.*
- *p induces an isomorphism $G/\Gamma_\infty \cong Q$.*

Proof. If Q is residually nilpotent, p induces a map $G/\Gamma_\infty \twoheadrightarrow Q$ between two residually nilpotent groups. Since $G \twoheadrightarrow G/\Gamma_\infty$ induces an isomorphism between the associated Lie rings, the statement for G can be deduced from the statement for G/Γ_∞. Thus, we can assume that G is residually nilpotent and, under this hypothesis, we need to show that p is an isomorphism if and only if $\mathcal{L}(p)$ is. Clearly, if p is an isomorphism, then $\mathcal{L}(p)$ is too. Conversely, if p, which is surjective, is not an isomorphism, then there is some non-trivial element x in its kernel. Since G is residually nilpotent, x induces a non-trivial class $\overline{x}$ in $\mathcal{L}(G)$, which is sent to 0 by $\mathcal{L}(p)$. This implies that $\mathcal{L}(p)$ is not injective, which concludes our proof. $\qquad\square$

1.3   Computing abelianisations from decompositions

Let us recall some classical tools for computing the abelianisation from some decomposition of a given group. The abelianisation functor $G \mapsto G^{\mathrm{ab}}$ is a left adjoint, hence right exact. In order to compute the abelianisation of an extension, one can say more:

Lemma 1.8 *Let $H \hookrightarrow G \twoheadrightarrow K$ be a short exact sequence of groups. It induces the following exact sequence of abelian groups:*

$$(H^{\mathrm{ab}})_K \to G^{\mathrm{ab}} \to K^{\mathrm{ab}} \to 0,$$

where coinvariants are taken with respect to the action of K on H^{ab} induced by conjugation in G.

Proof. The conjugation action of G on H^{ab} factors through $G/H = K$, hence $(H^{\mathrm{ab}})_G = (H^{\mathrm{ab}})_K$. Since we have an exact sequence $H^{\mathrm{ab}} \to G^{\mathrm{ab}} \to K^{\mathrm{ab}} \to 0$, it suffices to show that the morphism $H^{\mathrm{ab}} \to G^{\mathrm{ab}}$ factors through $(H^{\mathrm{ab}})_G$. It is equivariant with respect to the action of G induced by conjugation, which is obviously trivial on G^{ab}, whence the result. $\qquad\square$

For split exact sequences, we can say even more:

Lemma 1.9 *The abelianisation of a semidirect product $H \rtimes K$ is isomorphic to the product $(H^{\mathrm{ab}})_K \times K^{\mathrm{ab}}$, where $(H^{\mathrm{ab}})_K$ denotes the coinvariants of H^{ab} with respect to the induced action of K.*

Proof. As a consequence of the usual formula $[x, yz] = [x, y](x[x, z]x^{-1})$, one sees that the commutator subgroup $[H \rtimes K, H \rtimes K]$ is normally generated by $[H, H]$, $[H, K]$ and $[K, K]$. We can take the quotient by these three sets of relations successively: $(H \rtimes K)/[H, H]$ is isomorphic to $H^{\mathrm{ab}} \rtimes K$, then killing $[H, K]$ gives $(H^{\mathrm{ab}})_K \times K$ and finally, $((H^{\mathrm{ab}})_K \times K)/[K, K] \cong (H^{\mathrm{ab}})_K \times K^{\mathrm{ab}}$. $\qquad\square$

2 Strategy and first examples

In this section, we present some general ideas used to decide whether the lower central series stops or not. As a first example, we then apply these ideas to Artin groups.

2.1 Generation in degree one – first consequences

The Lie ring associated to the lower central series has the following fundamental property:

Proposition 2.1 *The Lie ring $\mathcal{L}(G)$ is generated in degree one. That is, it is generated by the abelianisation $\mathcal{L}_1(G) = G^{\mathrm{ab}}$ as a Lie algebra over $\mathbb{Z}$.*

Proof. It is a direct consequence of the definitions: the equality $\mathcal{L}_k(G) = [\mathcal{L}_1(G), \mathcal{L}_{k-1}(G)]$ is obtained directly from $\Gamma_k(G) = [\Gamma_1(G), \Gamma_{k-1}(G)]$, by passing to the appropriate quotients. $\qquad\square$

A first consequence of this is the following:

Corollary 2.2 *Let G be a group. If G^{ab} is cyclic, then $\Gamma_2 G = \Gamma_3 G$.*

Proof. Proposition 2.1 implies that the Lie ring $\mathcal{L}(G)$ is a quotient of the free Lie ring on G^{ab}. Since G^{ab} is cyclic, the latter is the abelian Lie ring consisting only of G^{ab}. As a consequence, $\Gamma_2 G/\Gamma_3 G = \mathcal{L}_2(G) = \{0\}$. $\qquad\square$

Example 2.3 (Braids) Directly from their usual presentations, one computes the abelianisation of the braid groups: $\mathbf{B}_n^{\mathrm{ab}} \cong \mathbb{Z}$ for $n \geqslant 2$. Thus $\Gamma_2(\mathbf{B}_n) = \Gamma_3(\mathbf{B}_n)$. This fact is originally due to [GL69], who proved it by different methods. This property of $\mathbf{B}_n$, which is also true for any quotient of $\mathbf{B}_n$ (such as the symmetric group $\mathfrak{S}_n$ for instance), may also be seen as a particular case of the computations for Artin groups below.

Example 2.4 (Knot groups) For any knot, the knot group has (infinite) cyclic abelianisation, thus its lower central series stops at Γ_2. This generalises readily to the enveloping groups of any connected quandles; see [BNS19, Prop. 3.3] for instance.

Example 2.5 (Automorphisms of free groups) Consider the automorphism group $\mathrm{Aut}(\mathbf{F}_n)$ of the free group on n letters. The kernel IA_n of the projection from $\mathrm{Aut}(\mathbf{F}_n)$ onto $\mathrm{Aut}(\mathbf{F}_n^{\mathrm{ab}}) \cong \mathrm{GL}_n(\mathbb{Z})$ is generated by the usual K_{ij} and K_{ijk} from [MKS04, §3.5], which are easily seen to be commutators of automorphisms. Thus $\mathrm{Aut}(\mathbf{F}_n)^{\mathrm{ab}} \cong \mathrm{GL}_n(\mathbb{Z})^{\mathrm{ab}}$. Whenever $n \geqslant 3$, this group is cyclic of order two, so the lower central series of $\mathrm{Aut}(\mathbf{F}_n)$ stops at Γ_2, and so does the one of $\mathrm{GL}_n(\mathbb{Z})$.

An easy generalisation of Corollary 2.2 is:

Corollary 2.6 *Let G be a group. Let S be a generating set of G^{ab}. Suppose that for each pair $(s,t) \in S^2$, we can find representatives $\tilde{s}, \tilde{t} \in G$ of s and t such that $\tilde{s}$ and $\tilde{t}$ commute. Then $\Gamma_2 G = \Gamma_3 G$.*

Proof. The Lie ring $\mathcal{L}(G)$ is generated by S. Moreover, the fact that $\left[\tilde{s}, \tilde{t}\right] = 1$ in G readily implies that $[s,t] = 0$ in $\mathcal{L}(G)$. Since the brackets $[s,t]$ for $(s,t) \in S^2$ generate $\mathcal{L}_2(G) = [\mathcal{L}_1(G), \mathcal{L}_1(G)]$, we see that $\Gamma_2 G / \Gamma_3 G = \mathcal{L}_2(G) = \{0\}$. In fact, $\mathcal{L}(G)$ is an abelian Lie ring, reduced only to $\mathcal{L}_1(G) = G^{\mathrm{ab}}$. $\qquad\square$

We have not made any effort to make the above corollary as general as possible. In particular, $\tilde{s}$ and $\tilde{t}$ may commute only up to an element of $\Gamma_3 G$, and the conclusion still holds. Also, one may think of similar statements showing that $\Gamma_3 G = \Gamma_4 G$, and so on. Weak as it may seem, our statement is already very useful. In particular, when applied to groups whose elements have a geometrical interpretation, it will often happen that $\tilde{s}$ and $\tilde{t}$ can be chosen *with disjoint support* (whatever this means, depending on the context – see for instance §4.1 for precise definitions in certain cases), which readily implies that they commute. We will sometimes need a more refined version of the above, but we will discuss it in each particular situation.

Example 2.7 (Automorphisms of $\mathbf{F}_2$) As an example of a case where Corollary 2.6 does not work, but the same kind of technique does apply, let us consider $\mathrm{Aut}(\mathbf{F}_2)$. We have mentioned that $\mathrm{Aut}(\mathbf{F}_n)^{\mathrm{ab}} \cong \mathrm{GL}_n(\mathbb{Z})^{\mathrm{ab}}$; see Example 2.5. For $n = 2$, this is no longer cyclic, but isomorphic to $(\mathbb{Z}/2)^2$, generated by the (equivalences classes of the) automorphisms σ and τ acting as follows (fixing free generators x and y of $\mathbf{F}_2$):

$$\sigma(x) = y \qquad \sigma(y) = x \qquad\qquad \tau(x) = x^{-1} \qquad \tau(y) = y.$$

It follows that $\mathcal{L}_2(\mathrm{Aut}(\mathbf{F}_2))$ is generated by the (equivalence class of the) automorphism $\iota = [\sigma, \tau]$ acting by $\iota(x) = x^{-1}$ and $\iota(y) = y^{-1}$. It is easy to check that ι commutes with both σ and τ, so $\mathcal{L}_3(\mathrm{Aut}(\mathbf{F}_2)) = 0$. Thus, the lower central series of $\mathrm{Aut}(\mathbf{F}_2)$ stops at Γ_3, as does that of $\mathrm{GL}_2(\mathbb{Z})$.

Let us spell out another useful consequence of Proposition 2.1:

Corollary 2.8 *Let G be a group and $d \geqslant 1$ be an integer. If $\mathcal{L}_k(G) = \Gamma_k G / \Gamma_{k+1} G$ is a d-torsion abelian group for some $k \geqslant 1$, then $\mathcal{L}_l(G) = \Gamma_l G / \Gamma_{l+1} G$ is too for all integers $l \geqslant k$.*

Proof. If an element $x \in \mathcal{L}(G)$ is of d-torsion, then for all $y \in \mathcal{L}(G)$, the bracket $[x, y]$ is too, because $d \cdot [x, y] = [d \cdot x, y] = 0$. Since $\mathcal{L}_{l+1}(G) = [\mathcal{L}_1(G), \mathcal{L}_l(G)]$, we get our result by induction on l. $\qquad\square$

Example 2.9 (Virtual and welded braids) Let G be a group such that $G^{\mathrm{ab}} \cong \mathbb{Z} \times (\mathbb{Z}/d)$, where the factors are generated, respectively, by u and v. Then $\mathcal{L}_2(G)$ is generated, as an abelian group, by $[u, v]$. Since $d \cdot [u, v] = [u, d \cdot v] = 0$, $\mathcal{L}_2(G)$ is of d-torsion, and then all the $\mathcal{L}_k(G)$, for $k \geqslant 2$, must be too. This applies, for instance, to the groups $\mathbf{vB}_2$ and $\mathbf{wB}_2$ (which are both isomorphic to $\mathbb{Z} * \mathbb{Z}/2$), with $d = 2$; see §4.

2.2 Artin groups

Let S be a set, and $M = (m_{s,t})_{s \in S}$ be a Coxeter matrix, i.e. a symmetric matrix with coefficients in $\mathbb{N} \cup \{\infty\}$, with $m_{s,s} = 1$, and $m_{s,t} \geqslant 2$ if $s \neq t$. Let A_M be the associated Artin group, defined by the presentation:

$$A_M = \left\langle S \;\middle|\; \underbrace{ststs\cdots}_{m_{s,t}} = \underbrace{tstst\cdots}_{m_{s,t}} \;\; (s, t \in S, \; m_{s,t} \neq \infty) \right\rangle.$$

Let us consider the graph $\mathcal{G}$ whose incidence matrix is M modulo 2. Namely, $\mathcal{G}$ is obtained by taking S as its set of vertices and by drawing an edge between s and t whenever $m_{s,t}$ is an odd integer.

Lemma 2.10 *The group A_M^{ab} is free abelian on the set $\pi_0(\mathcal{G})$ of connected components of $\mathcal{G}$.*

Proof. This is clear from the presentation: the relation between s and t becomes $\bar{s} = \bar{t}$ in A_M^{ab} if $m_{s,t}$ is odd, and becomes trivial if $m_{s,t}$ is even or $m_{s,t} = \infty$. $\quad\square$

Let us now study the lower central series of A_M. Suppose first that $\mathcal{G}$ is connected. Then A_M^{ab} is cyclic, and Corollary 2.2 applies: the lower central series of A_M stops at Γ_2. This holds in particular for the classical braid group; see Example 2.3. Now, if $\mathcal{G}$ has several connected components, we need to study the interactions between the corresponding generators of the Lie ring of A_M. The simplest case happens when all the even $m_{s,t}$ are equal to 2. Then A_M splits into the (restricted) direct product of the Artin groups corresponding to the connected components of our graph, thus $\mathcal{L}(A_M)$ is a direct sum of copies of $\mathbb{Z}$ (concentrated in degree one). Thus, we have obtained our first result:

Proposition 2.11 *If all the coefficients of M are finite, and are either odd or equal to 2, then the lower central series of A_M stops at Γ_2.*

In order to get a step further, we need to study more closely the interactions between the generators of the Lie ring of A_M corresponding to the connected components of $\mathcal{G}$.

Lemma 2.12 *Let $s, t \in S$. If $m_{s,t} = 2k$ for some integer k, then $k \cdot [\bar{s}, \bar{t}] = 0$ in $\mathcal{L}(A_M)$.*

Proof. If $m_{s,t} = 2k$, then $(st)^k = (ts)^k$ in A_M. We can write this relation as $(st)^k(ts)^{-k} = 1$. We recall that modulo $\Gamma_3(A_M)$, commutators commute with any other element. Hence, modulo $\Gamma_3(A_M)$, we have:

$$1 = (st)^k(ts)^{-k} = (st)^{k-1}[s,t](ts)^{-(k-1)} = [s,t](st)^{k-1}(ts)^{-(k-1)} = \cdots = [s,t]^k.$$

Thus, in $\mathcal{L}_2(A_M)$, we have $0 = k \cdot \overline{[s,t]} = k \cdot [\bar{s}, \bar{t}]$, as claimed. $\quad\square$

From this, we deduce that for $x, y \in \pi_0(\mathcal{G})$, we have $d_{x,y} \cdot [x, y] = 0$ in $\mathcal{L}_2(A_M)$, where $d_{x,x} = 1$ and $d_{x,y} = \gcd\left\{ \frac{m_{s,t}}{2} \mid s, t \in S, \ \bar{s} = x \text{ and } \bar{t} = y \right\}$ if $x \neq y$. In particular, this proves:

Proposition 2.13 *If all the coefficients of M are finite, then $\Gamma_2(A_M)/\Gamma_3(A_M)$ is of d-torsion, where $d = \mathrm{lcm}\{d_{x,y} \mid x, y \in \pi_0(\mathcal{G})\}$.*

Remark 2.14 Proposition 2.13 is more general than Proposition 2.11, which is recovered as a particular case where $d = 1$. In fact, for d to be equal to 1 (which implies that the lower central series stops at Γ_2), we do not need the even $m_{s,t}$ to be equal to 2, but only all the $d_{x,y}$ to be equal to 1.

Remark 2.15 Proposition 2.13 does not say anything when at least one of the $d_{x,y}$ is infinite. For instance, when all the $m_{s,t}$ are infinite, A_M is the free group on S, whose Lie ring is without torsion. More generally, Right-Angled Artin Groups (where all the $m_{s,t}$ are infinite or equal to 2) have torsion-free Lie rings [DK92].

3 Partitioned braids

This section is devoted to the study of lower central series of the group of *partitioned braids*; see Definition 3.2 below. Our main results are summarised in Theorem 3.5. The group of partitioned braids is a subgroup of $\mathbf{B}_n$, which has been studied notably in [Man97] and in [BGG17]. The former gave a presentation of this group, using the Reidemeister-Schreier method, which could be applied to this finite-index subgroup of Artin's braid group.

Remark 3.1 One could use the aforementioned presentation of the partitioned braid group [Man97] to get generators, a calculation of the abelianisation, and (with a little bit of work) the stable case in the study of the lower central series. However, we will avoid using this presentation

altogether, for several reasons. Firstly, we want our paper to be as self-contained as possible. Secondly, even if a presentation is of some help in the study of the lower central series of a group, there is only so much that one can deduce directly from it; in fact, not much more than a computation of the abelianisation, and that $\Gamma_2 = \Gamma_3$ when it holds. In particular, most of our non-stable results would not be simplified by using the presentation. Thirdly, and perhaps most importantly, to the best of our knowledge, nobody has written down a presentation of the other partitioned groups that we study later on. This could certainly be done using the Reidemeister-Schreier technique (at least when we have a presentation of the non-partitioned group), but this would require a fair amount of work, which we intend to avoid. And we will do so precisely by generalising proofs which do not depend on the use of Manfredini's presentation.

3.1 Reminders: braids

We recall that the standard generators σ_i and A_{ij} of the braid group $\mathbf{B}_n$ and of the pure braid group P_n respectively, are the braids drawn in Figure 3.1. In Figure 3.2 they also appear in a "bird's eye view" as paths of configurations.
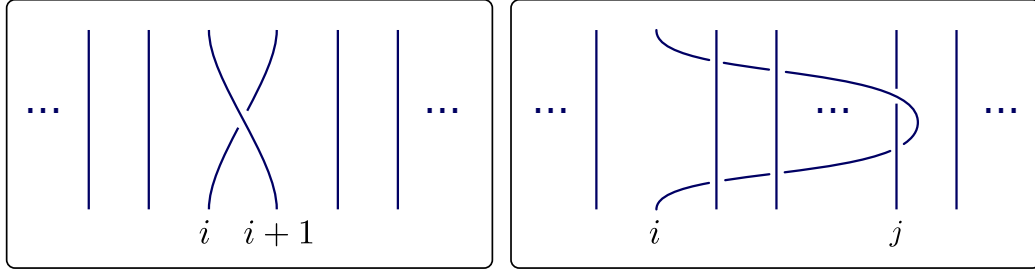

Figure 3.1   The standard generators σ_i and A_{ij} of, respectively, the braid group and of the pure braid group.

3.2 Basic theory of partitioned braids

For a partition $\lambda = (n_1, ..., n_l)$ of an integer n, we denote by $\mathfrak{S}_\lambda$ the subgroup $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ of the symmetric group $\mathfrak{S}_n$. Let us now consider the braid group $\mathbf{B}_n$ on n strands, and the usual projection $\pi : \mathbf{B}_n \twoheadrightarrow \mathfrak{S}_n$.

Definition 3.2 Let $n \geqslant 1$ be an integer, and let $\lambda = (n_1, ..., n_l)$ be a partition of n. The corresponding *partitioned braid group* (also referred to as the group of λ-*partitioned braids*) is:
$$\mathbf{B}_\lambda := \pi^{-1}(\mathfrak{S}_\lambda) = \pi^{-1}\left(\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}\right) \subseteq \mathbf{B}_n.$$

Lemma 3.3 *Let $\lambda = (n_1, ..., n_l)$ be a partition of an integer $n \geqslant 1$. Then $\mathbf{B}_\lambda$ is the subgroup of $\mathbf{B}_n$ generated by:*

- *The σ_α for $1 \leqslant \alpha \leqslant n$ such that α and $\alpha + 1$ are in the same block of λ.*
- *the $A_{\alpha\beta}$, for $1 \leqslant \alpha < \beta \leqslant n$ such that α and β do not belong to the same block of λ.*

Proof. Consider the subgroup G of $\mathbf{B}_n$ generated by these elements. Clearly, $G \subseteq \mathbf{B}_\lambda$, and we need to show that G contains $\mathbf{B}_\lambda$. First, we see that G contains $\mathbf{B}_{n_1} \times \cdots \times \mathbf{B}_{n_l}$, which is generated by the chosen σ_α. As a consequence, $\pi(G) = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$. Then, G also contains the $A_{\alpha\beta}$, for all $1 \leqslant \alpha < \beta \leqslant n$. Indeed, the $A_{\alpha\beta}$ missing in the list of the statement are the ones with α and β in the same block of λ, which are exactly the ones belonging to $\mathbf{B}_{n_1} \times \cdots \times \mathbf{B}_{n_l}$. Thus, G contains all of $\mathbf{P}_n$. Now, if β is any λ-partitioned braid, then $\pi(\beta) \in \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$, hence we can choose $g \in G$ such that $\pi(g) = \pi(\beta)$. Then $g^{-1}\beta$ is a pure braid, thus $g^{-1}\beta \in G$, which implies $\beta \in G$. $\square$

We now compute $\mathbf{B}_\lambda^{\mathrm{ab}}$, using the above, together with the split projections corresponding to forgetting blocks of strands. These projections can be seen as particular cases of the projections from Proposition 6.25 below, applied to braids on the disc.

Proposition 3.4 *Let $\lambda = (n_1, ..., n_l)$ be a partition of an integer $n \geqslant 1$. The abelianisation $\mathbf{B}_\lambda^{ab}$ is free abelian on the following basis:*

- *For each $i \in \{1, ..., l\}$ such that $n_i \geqslant 2$, one generator s_i: this is the common class of the σ_α for α and $\alpha + 1$ in the i-th block of λ.*
- *For each $i, j \in \{1, ..., l\}$ such that $i < j$, one generator a_{ij}: this is the common class of the $A_{\alpha\beta}$ for α in the i-th block of λ and β in the j-th one.*

Proof. The abelianisation $\mathbf{B}_\lambda^{ab}$ is generated by the classes of the generators from Lemma 3.3. Moreover, we note that for any i, all the σ_α with α and $\alpha + 1$ in the i-th block of λ (which exist only if $n_i \geqslant 2$) are conjugate to one another: for instance, if $n_1 \geqslant 3$, then $\sigma_2 = (\sigma_2\sigma_1)^{-1}\sigma_1(\sigma_2\sigma_1)$ is a conjugation relation inside $\mathbf{B}_\lambda$. Similarly, for each choice of $i < j$, all the $A_{\alpha\beta}$ with α in the i-th block and β in the j-th one are conjugate to one another. Thus, the family described in the statement is well-defined and generates $\mathbf{B}_\lambda^{ab}$. Let us show that it is linearly independent, by using the projections obtained by forgetting strands.

Suppose that $\sum k_i s_i + \sum k_{ij} a_{ij} = 0$ for some integers k_i ($i \leqslant l$) and k_{ij} ($i < j \leqslant l$). Let us fix i such that $n_i \geqslant 2$, and let us apply the canonical projection $\mathbf{B}_\lambda^{ab} \twoheadrightarrow \mathbf{B}_{n_i}^{ab} \cong \mathbb{Z}$ to the above relation. Since this projection kills all the generators save s_i, we get $k_i = 0$. This holds for all i, so we are left with the relation $\sum k_{ij} a_{ij} = 0$. To which, for any choice of $i < j$, we can apply the morphism $\mathbf{B}_\lambda^{ab} \twoheadrightarrow \mathbf{B}_{n_i, n_j}^{ab} \to \mathbf{B}_{n_i + n_j}^{ab} \cong \mathbb{Z}$. This kills all the a_{kl}, save a_{ij} (which is sent to 2), hence $k_{ij} = 0$, whence the result. $\square$

3.3 The lower central series

This section is devoted to the proof of the following result, which states exactly when the lower central series of the partitioned braid group stops.

Theorem 3.5 *Let $n \geqslant 1$ be an integer, let $\lambda = (n_1, ..., n_l)$ be a partition of n. The lower central series of the partitioned braid group $\mathbf{B}_\lambda$:*

- *stops at Γ_2 if $n_i \geqslant 3$ for all i, save at most two indices for which $n_i = 1$.*
- *does not stop in all the other cases, except for $\mathbf{B}_2 \cong \mathbb{Z}$.*

Proof. The first statement consists of Propositions 3.6, 3.8 and 3.9 (each of which is a generalisation of the previous one). The second one consists of the cases when λ has at least three blocks of size 1 (see Lemma 3.7), exactly one block of size 2 together with blocks of size at least 3 (see Proposition 3.11), at least two blocks of size 2 (see Corollary 3.14), or at least one block of size 1 and one block of size 2 (see Corollary 3.17). $\square$

3.3.1 The stable case: a disjoint support argument

Proposition 3.6 *Let $n \geqslant 1$ be an integer, let $\lambda = (n_1, ..., n_l)$ be a partition of n. Suppose that all the n_i are at least equal to 3. Then the lower central series of $\mathbf{B}_\lambda$ stops at Γ_2.*

Proof. Consider the generating set for $\mathbf{B}_\lambda^{ab}$ described in Corollary 3.4. For any pair of such generators, it is possible to find lifts in $\mathbf{B}_\lambda$ having disjoint support (as mapping classes of the punctured disc), and thus commuting – see Figure 3.2 for the three cases that may arise. Then, we can apply Corollary 2.6 to prove our claim. $\square$

3.3.2 Blocks of size 1

Lemma 3.7 *If at least three blocks of the partition λ are of size 1, then the lower central series of $\mathbf{B}_\lambda$ does not stop.*
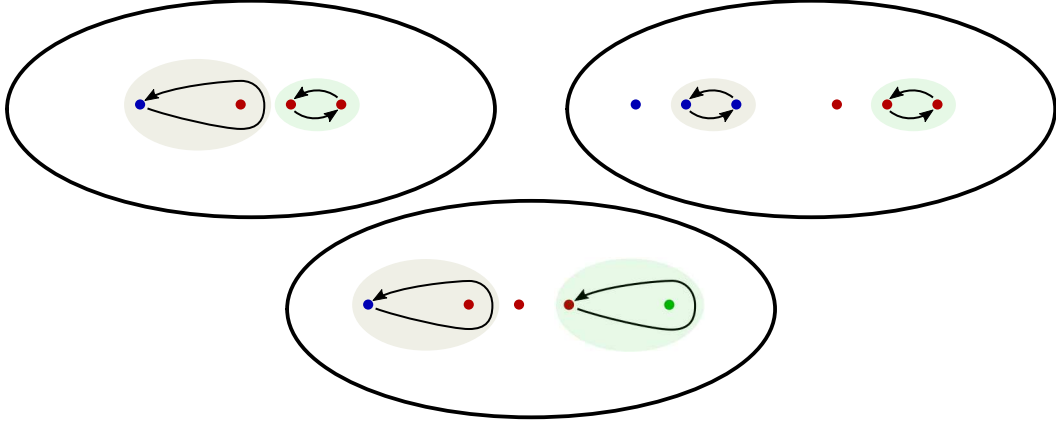
Figure 3.2   Choosing representatives with disjoint support for pairs of generators of the abelianisation of the partitioned braid group.

Proof. Under the hypothesis, there is a surjection $\mathbf{B}_\lambda \twoheadrightarrow \mathbf{B}_{1,1,1} \cong \mathbf{P}_3$ obtained by forgetting all the blocks save three blocks of size 1. The lower central series of $\mathbf{P}_3 \cong \mathbf{F}_2 \rtimes \mathbb{Z}$ does not stop, since it is an almost-direct product, and the lower central series of $\mathbf{F}_2$ does not stop; see for instance [FR85]. As a consequence, the one of $\mathbf{B}_\lambda$ does not either by Lemma 1.1. $\qquad\square$

Proposition 3.8 *Let $n \geqslant 1$ be an integer, let $\lambda = (1, n_2..., n_l)$ be a partition of n, with all the n_i for $i \geqslant 2$ at least equal to 3. Then the lower central series of $\mathbf{B}_\lambda$ stops at Γ_2.*

Proof. The case $l = 2$ (i.e. $\lambda = (1, m)$ for some integer $m \geqslant 3$) works exactly like the stable case of Proposition 3.6. Namely, the two generators of the abelianisation do have lifts with disjoint supports, so that $\Gamma_2(\mathbf{B}_{1,m}) = \Gamma_3(\mathbf{B}_{1,m})$ by Corollary 2.6. This however does not work for $l > 2$.

Let us denote by μ the partition $(n_2..., n_l)$ of $n - 1$, so that $\lambda = (1, \mu)$. We are going to show that $\mathbf{B}_\lambda / \Gamma_\infty$ is abelian, which implies $\Gamma_\infty = \Gamma_2$ for $\mathbf{B}_\lambda$.

It follows from Proposition 3.6 that $\Gamma_\infty = \Gamma_2$ for $\mathbf{B}_\mu$. As a consequence, the obvious morphism $\mathbf{B}_\mu \to \mathbf{B}_\lambda / \Gamma_\infty$ factors through $\mathbf{B}_\mu / \Gamma_2 \cong \mathbf{B}_\mu^{\mathrm{ab}}$, which implies in particular that modulo Γ_∞, we have $\sigma_\alpha \equiv \sigma_\alpha'$ and $A_{\alpha\beta} \equiv A_{\alpha'\beta'}$ if α and α' (resp. β and β') are in the same block of μ, and that the corresponding classes s_i and a_{ij} commute with one another. In the same way, we can deduce from the case of $\mathbf{B}_{1,m}$ for $m \geqslant 3$ treated above that the class of a generator $A_{1\alpha}$ depends only on the block of α. More precisely, we use the fact that the obvious morphism $\mathbf{B}_{1,n_i} \to \mathbf{B}_{1,\mu}/\Gamma_\infty$ factors through $\mathbf{B}_{1,n_i}^{\mathrm{ab}}$, where all the $A_{1\alpha}$ are identified. Moreover, the corresponding a_{1i} commutes with all the s_p and a_{pq} coming from $\mathbf{B}_\mu$, since they have lifts in $\mathbf{B}_\lambda$ with disjoint support.

We are left with showing that a_{1i} commutes with a_{1j} when $i < j$. This uses Lemma 3.10 below. Let us choose any α in the i-th block and any β in the j-th one, and consider the morphism $\mathbf{P}_3 \to \mathbf{B}_\lambda$ induced by $1 \mapsto 1$, $2 \mapsto \alpha$ and $3 \mapsto \beta$. If we compose it with the projection onto $\mathbf{B}_\lambda / \Gamma_\infty$, we get a morphism f sending A_{12} to a_{1i}, A_{13} to a_{1j}, and A_{23} to a_{ij}. Since a_{ij} commutes with a_{1j}, f sends $[A_{13}, A_{23}]$ to 1. Lemma 3.10 then implies that it factors through $\mathbf{P}_3^{\mathrm{ab}}$, so that all the elements in its image commute, including a_{1i} and a_{1j}. This finishes the proof that $\mathbf{B}_\lambda / \Gamma_\infty$ is abelian, whence equal to $\mathbf{B}_\lambda / \Gamma_2$. $\qquad\square$

Proposition 3.9 *Let $n \geqslant 1$ be an integer, let $\lambda = (1, 1, n_3..., n_l)$ be a partition of n, with all the n_i for $i \geqslant 3$ at least equal to 3. Then the lower central series of $\mathbf{B}_\lambda$ stops at Γ_2.*

Proof. Let μ denote the partition $(n_3..., n_l)$ of $n - 2$, so that $\lambda = (1, 1, \mu)$. It follows from Proposition 3.8 that $\Gamma_\infty = \Gamma_2$ for $\mathbf{B}_{1,\mu}$, hence both the obvious maps $\mathbf{B}_{1,\mu} \to \mathbf{B}_\lambda / \Gamma_\infty$ factor through $\mathbf{B}_{1,\mu}^{\mathrm{ab}}$. From this, we deduce that $\mathbf{B}_\lambda / \Gamma_\infty$ is generated by elements s_i ($3 \leqslant i \leqslant l$) and a_{ij} ($1 \leqslant i < j \leqslant l$), and that the a_{1j} (resp. the a_{2j}), the s_p ($p \geqslant 1$) and the a_{pq} ($q > p \geqslant 3$) commute with one another. Moreover, for all $i, j \geqslant 3$, a_{1i} and a_{2j} have lifts with disjoint support in $\mathbf{B}_\lambda$, so they commute (even when $i = j$).

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We are left with showing that the class a_{12} of A_{12} commutes with all the other generators. A disjoint support argument shows that it commutes with the s_k and a_{kl} for $l > k \geqslant 3$. Now, for $i \geqslant 3$, let us choose α in the i-th block, and let us consider the morphism $\mathbf{P}_3 \to \mathbf{B}_\lambda$ induced by $1 \mapsto 1$, $2 \mapsto 2$ and $3 \mapsto \alpha$. If we compose it with the projection onto $\mathbf{B}_\lambda / \Gamma_\infty$, we get a morphism f sending A_{12} to a_{12}, A_{13} to a_{1i}, and A_{23} to a_{2i}. Since a_{1i} commutes with a_{2i}, f sends $[A_{13}, A_{23}]$ to 1. Lemma 3.10 then implies that it factors through $\mathbf{P}_3^{\mathrm{ab}}$, so that all the elements in its image commute, showing that a_{12} commutes with a_{1i} and with a_{2i}. Thus, we have proved that $\mathbf{B}_\lambda / \Gamma_\infty$ is abelian, whence $\Gamma_\infty(\mathbf{B}_\lambda) = \Gamma_2(\mathbf{B}_\lambda)$. $\qquad\square$

Lemma 3.10 *The quotient of* $\mathbf{P}_3$ *by the relation* $[A_{13}, A_{23}] = 1$ *is* $\mathbf{P}_3^{\mathrm{ab}} \cong \mathbb{Z}^3$.

Proof. Let N be the normal closure of $[A_{13}, A_{23}]$ in $\mathbf{P}_3$. We want to show that $N = \Gamma_2(\mathbf{P}_3)$. Clearly, $N \subseteq \Gamma_2(\mathbf{P}_3)$. To show the converse inclusion, we need to show that $\mathbf{P}_3/N$ is abelian. We check that the relations $A_{12}A_{13}A_{12}^{-1} = A_{23}^{-1}A_{13}A_{23}$ and $A_{12}^{-1}A_{23}A_{12} = A_{13}A_{23}A_{13}^{-1}$ hold in $\mathbf{P}_3$. As a consequence, modulo the relation $[A_{13}, A_{23}] = 1$ (that is, modulo N), we get $A_{12}A_{13}A_{12}^{-1} \equiv A_{13}$ and $A_{12}^{-1}A_{23}A_{12} \equiv A_{23}$. Thus A_{12}, A_{13} and A_{23} commute modulo N and therefore $\mathbf{P}_3/N$ is abelian. $\qquad\square$

3.3.3 Blocks of size 2

When there is exactly one block of size 2 and no block of size 1, we get a complete description of the quotient of $\mathbf{B}_\lambda$ by its residue:

Proposition 3.11 *Let* $n \geqslant 1$ *be an integer, let* $\lambda = (2, n_2..., n_l)$ *be a partition of* n*, with all the* n_i *at least equal to 3. Then* $\mathbf{B}_\lambda / \Gamma_\infty$ *decomposes as a direct product of* $l(l-1)/2$ *copies of* $\mathbb{Z}$ *with* $\mathbb{Z}^{2(l-1)} \rtimes \mathbb{Z}$*, where* $\mathbb{Z}$ *acts via the involution exchanging the elements* e_{2i} *and* e_{2i+1} *of a basis of* $\mathbb{Z}^{2(l-1)}$*. In particular, if* $l \geqslant 2$*, then the lower central series of* $\mathbf{B}_\lambda$ *does not stop.*

Proof. Let μ denote the partition $(n_2, ..., n_l)$ of $n - 2$, so that $\lambda = (2, \mu)$. Then the canonical projection $\mathbf{B}_{2,\mu} \twoheadrightarrow \mathfrak{S}_{2,\mu} \twoheadrightarrow \mathfrak{S}_2$ has $\mathbf{B}_{1,1,\mu}$ as its kernel. Moreover, $\Gamma_\infty(\mathbf{B}_{2,\mu})$ contains $\Gamma_\infty(\mathbf{B}_{1,1,\mu})$, which is equal to $\Gamma_2(\mathbf{B}_{1,1,\mu})$ by Proposition 3.9. We show that these are in fact equal. In order to do this, it is enough to show that $\mathbf{B}_{2,\mu}/\Gamma_2(\mathbf{B}_{1,1,\mu})$ is residually nilpotent. In fact, we are going to compute it completely.

First, let us remark that it makes sense to consider this quotient: $\Gamma_2(\mathbf{B}_{1,1,\mu})$ is a characteristic subgroup of $\mathbf{B}_{1,1,\mu}$, which is normal (of index 2) in $\mathbf{B}_{2,\mu}$, hence it is a normal subgroup of $\mathbf{B}_{2,\mu}$.

Next, we can write $\mathbf{B}_{2,\mu}/\Gamma_2(\mathbf{B}_{1,1,\mu})$ as an extension:

$$\mathbf{B}_{1,1,\mu}^{\mathrm{ab}} \lhook\joinrel\longrightarrow \mathbf{B}_{2,\mu}/\Gamma_2(\mathbf{B}_{1,1,\mu}) \longrightarrow\joinrel\twoheadrightarrow \mathfrak{S}_2. \qquad (3.1)$$

We can use the method from §A.3 to get a presentation of $G := \mathbf{B}_{2,\mu}/\Gamma_2(\mathbf{B}_{1,1,\mu})$. Namely, we have a presentation of the kernel: $\mathbf{B}_{1,1,\mu}^{\mathrm{ab}}$ is free abelian on the s_i and the a_{ij} indexed by the blocks of $(1, 1, \mu)$. We also know the action of $\mathfrak{S}_2$ on $\mathbf{B}_{1,1,\mu}^{\mathrm{ab}}$ induced by conjugation by σ_1 in $\mathbf{B}_{2,\mu}$: it exchanges the a_{1j} with the a_{2j} with $j \geqslant 3$ and it acts trivially on all the other generators. Finally, we can lift the only relation defining $\mathfrak{S}_2$ to $\sigma_1^2 = A_{12}$ in $\mathbf{B}_{2,\mu}$. As a consequence, we get the presentation:

$$G = \left\langle \begin{array}{l} s, \\ s_i \ (3 \leqslant i \leqslant l+1), \\ a_{ij} \ (1 \leqslant i < j \leqslant l+1). \end{array} \middle| \begin{array}{ll} \forall i, j, p, q, u, v, & [s_i, s_j] = [s_i, a_{pq}] = [a_{pq}, a_{uv}] = 1, \\ \forall i \geqslant 1, & [s, s_i] = 1, \\ \forall j > i \geqslant 3, & [s, a_{ij}] = 1, \\ \forall j \geqslant 3, & s a_{1j} s^{-1} = a_{2j} \text{ and } s a_{2j} s^{-1} = a_{1j}, \\ & s^2 = a_{12}. \end{array} \right\rangle$$

Then one can deduce from this presentation that G decomposes as $\mathbb{Z}^N \times (\mathbb{Z}^{2(l-1)} \rtimes \mathbb{Z})$ where the first factor is free abelian on the s_i $(i \geqslant 3)$ and the a_{ij} $(j > i \geqslant 3)$ (hence $N = l(l-1)/2$), and the action of $\mathbb{Z}$ (free on s) on $\mathbb{Z}^{2(l-1)}$ (free abelian on the a_{1j} and the a_{2j}) is given by s exchanging the a_{1j} and the a_{2j}. Checking that this holds is a matter of writing the appropriate split projections from the presentations of G_l and its factors.

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Finally, the decomposition of G allows us to apply Proposition B.10 to compute its lower central series. Namely, we apply it to $A = \langle a_{1j}, a_{2j} \rangle_{j \geqslant 3}$ (which is free abelian on these generators) endowed with the involution τ exchanging a_{1j} with a_{2j} for all j. Then $V = \mathrm{Im}(\tau - 1)$ is the free abelian subgroup generated by the $a_{1j} - a_{2j}$, and for $k \geqslant 2$, we have $\Gamma_k(G_l) = 2^{k-1}V$. In particular, this lower central series does not stop. However its intersection is trivial: the group G_l is residually nilpotent, which implies that $\Gamma_\infty(\mathbf{B}_{2,\mu}) = \Gamma_\infty(\mathbf{B}_{1,1,\mu})$, and finishes our proof. $\qquad\square$

Remark 3.12 The Lie ring of G (which identifies with the Lie ring of $\mathbf{B}_{2,\mu}$) can be completely computed, using Corollary B.11. Namely, it identifies with $\mathbb{Z}^N \times (L \rtimes \mathbb{Z})$, where $L = \mathbb{Z}^{l-1} \oplus (\mathbb{Z}/2)^{l-1} \oplus (\mathbb{Z}/2)^{l-1} \oplus \cdots$ and the action of the generator t of $\mathbb{Z}$ on L is via the degree-one map $\mathbb{Z}^{l-1} \twoheadrightarrow (\mathbb{Z}/2)^{l-1} \cong (\mathbb{Z}/2)^{l-1} \cong \cdots$. In other words, as a Lie ring, $\mathcal{L}(\mathbf{B}_{2,\mu})$ admits the presentation via generators $t, X_1, ..., X_{l-1}, Y_1, ..., Y_N$ and relations:

$$\begin{cases} [Y_i, Y_j] = [Y_i, X_k] = [Y_i, t] = [X_k, X_l] = 0, \\ 2[t, X_i] = 0. \end{cases}$$

Proposition 3.13 *The lower central series of* $\mathbf{B}_{2,2}$ *does not stop.*

Proof. Since $\mathbf{B}_{2,2}$ surjects onto $\mathbf{B}_{2,2}(\mathbb{S}^2)$, this is a direct consequence of Proposition 6.53, by an application of Lemma 1.1. Alternatively, one can adapt the proof of Proposition 6.53 to this case, getting that:

$$\mathbf{B}_{2,2}/\langle A_{12}, A_{34}, \Gamma_2(\mathbf{P}_4) \rangle \cong (\mathbb{Z}^2)^{\otimes 2} \rtimes (\mathfrak{S}_2)^2,$$

where $\mathfrak{S}_2$ acts on $\mathbb{Z}^2$ by permutation of the factors. Then the methods of the appendix can be used to compute completely the lower central series of this group. $\qquad\square$

Corollary 3.14 *If at least two blocks of the partition* λ *are of size* 2, *then the lower central series of* $\mathbf{B}_\lambda$ *does not stop.*

Proof. Under the hypothesis, there is a surjection $\mathbf{B}_\lambda \twoheadrightarrow \mathbf{B}_{2,2}$. Thus, this corollary is obtained from a direct application of Lemma 1.1. $\qquad\square$

3.3.4 Blocks of size 1 and 2: study of $\mathbf{B}_{1,2}$

We use that $\mathbf{B}_{1,2}$ is isomorphic to the Artin group of type $\mathbf{B}_2$, a classical fact of which we give a proof, for the sake of completeness.

Lemma 3.15 *The group* $\mathbf{B}_{1,2}$ *is isomorphic to the Artin group* $G = \langle \sigma, x \mid (\sigma x)^2 = (x\sigma)^2 \rangle$. *As a consequence, it is residually nilpotent, but not nilpotent. In particular, its lower central series does not stop.*

Proof. On the one hand, we can re-write the presentation of G as:

$$G = \langle \sigma, x, y \mid (\sigma x)^2 = (x\sigma)^2, \ y = \sigma x \sigma^{-1} \rangle.$$

Then, modulo the second relation (which we conveniently rewrite $y\sigma = \sigma x$), the first one is equivalent to $(y\sigma)^2 = x(y\sigma)\sigma$, and in turn to $\sigma y \sigma^{-1} = y^{-1}xy$. Thus:

$$G = \langle \sigma, x, y \mid \sigma x \sigma^{-1} = y, \ \sigma y \sigma^{-1} = y^{-1}xy \rangle.$$

On the other hand, the projection $\mathbf{B}_{1,2} \twoheadrightarrow \mathbf{B}_2 \cong \mathbb{Z}$ splits, and its kernel identifies with the fundamental group of the disc minus two points, which is free on two generators x and y. In fact, the corresponding action of $\mathbf{B}_2 \cong \langle \sigma \rangle$ on $\mathbf{F}_2$ is the usual Artin action. Which means exactly that the above relations are true in $\mathbf{B}_{1,2}$. Whence a surjection of G onto $\mathbf{B}_{1,2}$, which induces a diagram with (split) short exact rows:

$$\begin{array}{ccccc} \langle x, y \rangle & \hookrightarrow & G & \twoheadrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ F_2 & \hookrightarrow & \mathbf{B}_{1,2} & \twoheadrightarrow & \mathbf{B}_2 \cong \mathbb{Z}. \end{array}$$

Using the freeness of their target, one sees that the left and right vertical maps must be isomorphisms, hence so is the middle map, by the Five Lemma. The rest of the statement is then a reformulation of Proposition B.20. □

Remark 3.16 More precision is given by Proposition B.21, which describes the Lie ring of the group G (hence of $\mathbf{B}_{1,2}$). If, on the contrary, one wishes only to see that the lower central series of G does not stop, one can argue directly from $G \twoheadrightarrow \mathbb{Z} * (\mathbb{Z}/2) \twoheadrightarrow (\mathbb{Z}/2) * (\mathbb{Z}/2) \cong \mathbb{Z} \rtimes (\mathbb{Z}/2)$ and the explicit computation of the lower central series of $\mathbb{Z} \rtimes (\mathbb{Z}/2)$, see §B.3.2.

Corollary 3.17 *If the partition λ has both a block of size 1 and a block of size 2, then the lower central series of $\mathbf{B}_\lambda$ does not stop.*

Proof. There is a surjection $\mathbf{B}_\lambda \twoheadrightarrow \mathbf{B}_{1,2}$, to which we can apply Lemma 1.1. □

4 Virtual and welded braids

In this section, we study the lower central series of the *virtual*, *welded* and *extended welded* braid groups, which are generalisations of the classical braid groups. Presentations of these groups may be found in §A.1.1. Our results for each of them are given, respectively, in Propositions 4.9, 4.11 and 4.16. We also study the lower central series for the *partitioned* versions of each of these groups; the corresponding results are summed up as Theorem 4.22.

4.1 Notion of support of an element

In order to apply disjoint support arguments, we first need to discuss the notion of support for elements of our groups. We will use the following notion of *support* of an automorphism of $\mathbf{F}_n$.

Definition 4.1 Let $\mathbf{F}_n$ be the free group on n (fixed) generators $x_1, ..., x_n$, and let $\varphi \in \mathrm{Aut}(\mathbf{F}_n)$. The *support* $\mathrm{Supp}(\varphi)$ of φ is:

$$\mathrm{Supp}(\varphi) := \{i \in \{1, \ldots, n\} \mid \varphi(x_i) \neq x_i \text{ or } x_i \text{ appears in some } \varphi(x_j) \text{ for } j \neq i\},$$

where x_i is said to appear in $w \in \mathbf{F}_n$ if $x_i^{\pm 1}$ is a letter in the reduced expression of w.

Example 4.2 With the notations from section A.1.4, we have:

- $\mathrm{Supp}(\rho_\alpha) = \{\alpha\}$
- $\mathrm{Supp}(\sigma_\alpha) = \mathrm{Supp}(\tau_\alpha) = \{\alpha, \alpha+1\}$
- $\mathrm{Supp}(\chi_{\alpha\beta}) = \{\alpha, \beta\}$

Fact 4.3 *If φ and ψ are automorphisms of $\mathbf{F}_n$ with disjoint support, then they commute.*

Proof. Observing that $\varphi(x_i) = x_i$ for $i \notin \mathrm{Supp}(\varphi)$, and that $\varphi(x_i)$ is a word in $\{x_j \mid j \in \mathrm{Supp}(\varphi)\}$ for $i \in \mathrm{Supp}(\varphi)$, it is easy to see that

$$\psi(\varphi(x_i)) = \varphi(\psi(x_i)) = \begin{cases} \varphi(x_i) & i \in \mathrm{Supp}(\varphi) \\ \psi(x_i) & i \in \mathrm{Supp}(\psi) \\ x_i & i \notin \mathrm{Supp}(\varphi) \cup \mathrm{Supp}(\psi), \end{cases}$$

for all $i \in \{1, \ldots, n\}$. □

Welded braid groups $\mathbf{wB}_n$ and extended welded braid groups $\widetilde{\mathbf{w}}\mathbf{B}_n$ are naturally subgroups of $\mathrm{Aut}(\mathbf{F}_n)$, so this defines the support of an element of any of these groups. On the other hand, virtual braid groups $\mathbf{vB}_n$ do not naturally embed into $\mathrm{Aut}(\mathbf{F}_n)$, so we define the support of their elements differently, using the diagrammatic description (§A.1.2) of $\mathbf{vB}_n$ as a group of equivalence classes of n-strand braid diagrams with both classical and virtual crossings.

Definition 4.4 Let $\varphi \in \mathbf{vB}_n$. The *support* $\mathrm{Supp}(\varphi) \subseteq \{1, \ldots, n\}$ of φ is the set of $i \in \{1, \ldots, n\}$ such that either

- i is not fixed by the permutation of $\{1, \ldots, n\}$ induced by φ, or
- in every n-strand diagram representing φ, there is a classical crossing on the i-th strand.

Welded braid groups $\mathbf{wB}_n$ also admit a similar diagrammatic description (see §A.1.2), so one may define $\mathrm{Supp}(\varphi)$, for $\varphi \in \mathbf{wB}_n \subseteq \mathrm{Aut}(\mathbf{F}_n)$, either as in Definition 4.1 or as in Definition 4.4.

Lemma 4.5 *These two definitions of* $\mathrm{Supp}(\varphi)$ *for* $\varphi \in \mathbf{wB}_n \subseteq \mathrm{Aut}(\mathbf{F}_n)$ *agree.*

Proof. According to Definition 4.4, we have $i \notin \mathrm{Supp}(\varphi)$ if and only if i is fixed by the permutation of $\{1, \ldots, n\}$ induced by φ and there is an n-strand diagram representing φ with no classical crossings on the i-th strand. The first condition is equivalent to saying that $\varphi(x_i)$ is conjugate to x_i. The second condition splits into two statements: there is an n-strand diagram of φ with no classical crossings involving the i-th strand as the *upper* strand and no classical crossings involving the i-th strand as the *lower* strand. These two statements corresponds to saying that no x_j (for $j \neq i$) appears in $\varphi(x_i)$ and that x_i does not appear in any $\varphi(x_j)$ (for $j \neq i$). Combining the first of these statements with the condition that $\varphi(x_i)$ is conjugate to x_i, we obtain the condition that $\varphi(x_i) = x_i$. Hence $i \notin \mathrm{Supp}(\varphi)$ if and only if $\varphi(x_i) = x_i$ and x_i does not appear in any $\varphi(x_j)$ (for $j \neq i$). This is precisely Definition 4.1. $\square$

Remark 4.6 Similarly, *extended* welded braid groups $\widetilde{\mathbf{w}}\mathbf{B}_n$ also admit a diagrammatic description (see again §A.1.2), and so in this case one also has two possible definitions of $\mathrm{Supp}(\varphi)$ using either Definition 4.4 or Definition 4.1. As in Lemma 4.5 for the non-extended case, these two definitions agree.

Fact 4.7 *If* φ *and* ψ *are elements of* $\mathbf{vB}_n$ *with disjoint support, then they commute.*

Proof. It will suffice to show that φ and ψ commute after conjugating them both by the same element of $\mathbf{vB}_n$. By conjugating with an appropriate element (involving only virtual crossings), we may therefore arrange that $\mathrm{Supp}(\varphi) = \{1, \ldots, r\}$ and $\mathrm{Supp}(\psi) = \{r + 1, \ldots, r + s\}$ for some $r + s \leqslant n$. By the virtual Reidemeister moves (specifically the *detour moves*), we may then arrange that we have diagrams D_φ, D_ψ for φ, ψ such that

$$D_\varphi = D'_\varphi \otimes \mathbb{I}_{n-r} \qquad \text{and} \qquad D_\psi = \mathbb{I}_r \otimes D'_\psi \otimes \mathbb{I}_{n-r-s},$$

where $\mathbb{I}_k$ denotes the trivial k-strand diagram and $\otimes$ denotes placing virtual braid diagrams side-by-side. By "vertically sliding the subdiagrams D'_φ and D'_ψ", we see that

$$\begin{aligned}
D_\varphi \circ D_\psi &= \left(D'_\varphi \otimes \mathbb{I}_s \otimes \mathbb{I}_{n-r-s} \right) \circ \left(\mathbb{I}_r \otimes D'_\psi \otimes \mathbb{I}_{n-r-s} \right) \\
&= \left(\mathbb{I}_r \otimes D'_\psi \otimes \mathbb{I}_{n-r-s} \right) \circ \left(D'_\varphi \otimes \mathbb{I}_s \otimes \mathbb{I}_{n-r-s} \right) \\
&= D_\psi \circ D_\varphi. \qquad \qquad \square
\end{aligned}$$

Remark 4.8 There is another interpretation of the notion of *support* in the case of *welded* and *extended welded* braids, where we view elements as loops of configurations of unlinked, unknotted circles. In this interpretation the notion of support is more visually obvious, without having to explicitly invoke the virtual and welded Reidemeister moves. For example, Figures 4.1 and 4.2 depict the same pair of elements, $\chi_{\beta,\alpha'}$ and σ_α, representing generators x_{ji} and s_i of the abelianisation of the welded braid group. In each case the supports are coloured blue and red respectively: in the former case these are (disjoint) compact subsets of $\mathbb{R}^3$, in the latter case they are (disjoint) subcollections of strands.

4.2 Virtual braids

The group $\mathbf{vB}_n$ of virtual braids is a quotient of $\mathbf{B}_n * \mathfrak{S}_n$, obtained by adding some mixed relations between the generators of $\mathbf{B}_n$ and those of $\mathfrak{S}_n$; see the presentation (A.1). These mixed relations
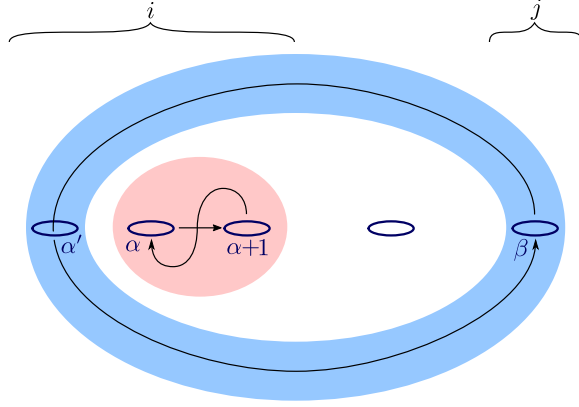
Figure 4.1   The supports of σ_α and $\chi_{\beta,\alpha'}$ interpreted as loops of configurations of unlinks.
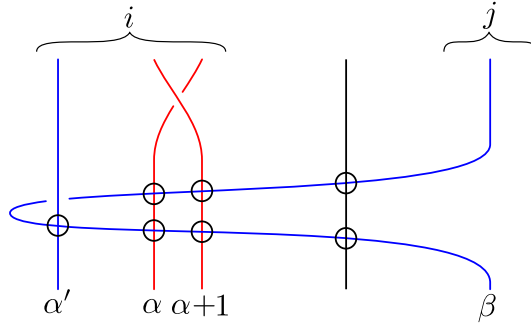

Figure 4.2   The supports of σ_α and $\chi_{\beta,\alpha'}$ interpreted diagrammatically.

correspond in the diagrammatic setting to the mixed Reidemeister-III moves described on the left-hand side of Figure A.3.

Let us remark that we can compute $\mathbf{wB}_n^{\mathrm{ab}}$ easily, using the presentation of Proposition A.2. Namely, for $n \geqslant 2$, $\mathbf{wB}_n^{\mathrm{ab}} \cong \mathbb{Z} \oplus (\mathbb{Z}/2)$, where the first factor is generated by the common class σ of the braid generators σ_i, and the second one is generated by the common class τ of the symmetric generators τ_i.

Proposition 4.9 *The lower central series of the virtual braid group $\mathbf{vB}_n$:*

- stops at Γ_2 *if $n \geqslant 4$.*
- does not stop *if $n = 2, 3$.*

Moreover, $\mathbf{vB}_2$ is residually nilpotent, whereas $\mathbf{vB}_3$ is not.

Proof. From the presentation of the virtual braid group $\mathbf{vB}_n$ of Proposition A.2, we deduce that in $\mathbf{vB}_n^{\mathrm{ab}}$ all the σ_i have the same class σ and all the τ_i have the same class τ.

Case 1: $n \geqslant 4$. Then σ_1 and τ_3 represent respectively σ and τ, and they commute, since they have disjoint support ; see Figure 4.3 for an illustration. Hence we can apply Corollary 2.6 to conclude that $\Gamma_2(\mathbf{vB}_n) = \Gamma_3(\mathbf{vB}_n)$.

Case 2: $n = 2$. The group $\mathbf{vB}_2 \cong \mathbb{Z} * \mathbb{Z}/2$ is residually nilpotent, but not nilpotent; see Proposition B.19. In particular, its lower central series does not stop.

Case 3: $n = 3$. One can see directly from the presentation of $\mathbf{vB}_3$ that there is a well-defined surjection $\pi : \mathbf{vB}_3 \twoheadrightarrow \mathbf{vB}_2$, sending σ_1 and σ_2 to σ_1, and τ_1 and τ_2 to τ_1. This induces a surjection $\mathcal{L}(\pi) : \mathcal{L}(\mathbf{vB}_3) \twoheadrightarrow \mathcal{L}(\mathbf{vB}_2)$, hence the lower central series of $\mathbf{vB}_3$ does not stop by Lemma 1.1.

Finally, the canonical map $\mathfrak{S}_3 \to \mathbf{vB}_3$ sending τ_i to τ_i ($i \in \{1,2\}$) splits, is thus injective, and $\mathfrak{S}_3$ is not residually nilpotent. This implies that $\mathbf{vB}_3$ is not residually nilpotent. $\qquad\square$

We can describe precisely the behaviour of the lower central series of $\mathbf{vB}_3$:

Proposition 4.10 *The surjection $\pi : \mathbf{vB}_3 \twoheadrightarrow \mathbf{vB}_2$ described in the proof of Proposition 4.9 induces an isomorphism $\mathbf{vB}_3/\Gamma_\infty \cong \mathbf{vB}_2$. In particular, $\mathcal{L}(\pi)$ is an isomorphism between the associated Lie rings.*

Proof. We first show that $\mathcal{L}(\pi)$ is an isomorphism. The surjection π splits, a splitting ι being given by $\sigma_1 \mapsto \sigma_1$ and $\tau_1 \mapsto \tau_1$ and corresponds to adding a third strand. This splitting induces a surjection between abelianisations. Since the Lie rings associated to lower central series are generated by their degree one, this implies that $\mathcal{L}(\iota)$ is surjective. Thus, $\mathcal{L}(\pi)$ and $\mathcal{L}(\iota)$ are isomorphisms $\mathcal{L}(\mathbf{vB}_3) \cong \mathcal{L}(\mathbf{vB}_2)$. Now, since $\mathbf{vB}_2 \cong \mathbb{Z} * \mathbb{Z}/2$ is residually nilpotent, the result follows directly from Lemma 1.7. $\square$
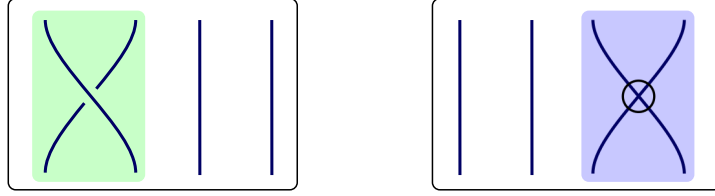

Figure 4.3   Disjoint-support representatives for the two generators of the abelianisation of the virtual braid group $\mathbf{vB}_n$ for $n \geqslant 4$. The supports are shaded in green and blue respectively.

4.3   Welded braids

The group $\mathbf{wB}_n$ of welded braids, also known as the group $\mathbf{LB}_n$ of loop braids, is a quotient of $\mathbf{vB}_n$, obtained by adding the relations $\sigma_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \sigma_{i+1}$; see the presentation (A.2). This relation corresponds to the welded Reidemeister-III move described on the right-hand side of Figure A.3. This allows us to deduce some results about the lower central series of $\mathbf{wB}_n$ from their counterparts for $\mathbf{vB}_n$. As a first example of this, note that $\mathbf{wB}_n^{\mathrm{ab}} \cong \mathbf{vB}_n \cong \mathbb{Z} \oplus (\mathbb{Z}/2)$.

Proposition 4.11 *The lower central series of the welded braid group $\mathbf{wB}_n$:*

- *stops at Γ_2 if $n \geqslant 4$.*
- *does not stop if $n = 2, 3$.*

Moreover, $\mathbf{wB}_2$ is residually nilpotent, whereas $\mathbf{wB}_3$ is not.

Proof. **Case 1:** $n \geqslant 4$. That $\Gamma_2(\mathbf{vB}_n) = \Gamma_3(\mathbf{vB}_n)$ directly implies $\Gamma_2(\mathbf{wB}_n) = \Gamma_3(\mathbf{wB}_n)$. In fact, one can easily see that our disjoint support argument for $\mathbf{vB}_n$ can be used in $\mathbf{wB}_n$ too. See Figure 4.4 for an illustration from the point of view of "loop braids"; alternatively, see Figure 4.3 again, interpreted now as a diagram of welded braids instead of virtual braids.

Case 2: $n = 2$. There is no additional relation: $\mathbf{wB}_2 \cong \mathbf{vB}_2 \cong \mathbb{Z} * \mathbb{Z}/2$.

Case 3: $n = 3$. We treat this tricky case as a separate result. Namely, Proposition 4.12 below gives a surjection from $\mathbf{wB}_3$ onto $\mathbb{Z}/2 * \mathbb{Z}/2$, whose lower central series does not stop by Proposition B.16. The result thus follows from Lemma 1.1. $\square$

The group $\mathbf{wB}_3$ is a non-trivial quotient of $\mathbf{vB}_3$, and our result on $\mathbf{vB}_3$ cannot be transferred to $\mathbf{wB}_3$, nor can we adapt directly the reasoning: the surjection $\mathbf{vB}_3 \twoheadrightarrow \mathbf{vB}_2 \cong \mathbf{wB}_2$ does not factor through the quotient by our additional relation, leaving us with no obvious surjection from $\mathbf{wB}_3$ onto $\mathbf{wB}_2$. However, we are still able to compute $\mathbf{wB}_3/\Gamma_\infty$, and the situation is very similar to the case of virtual braids. In particular, the lower central series does not stop, but $\mathbf{wB}_3$ is not residually nilpotent.
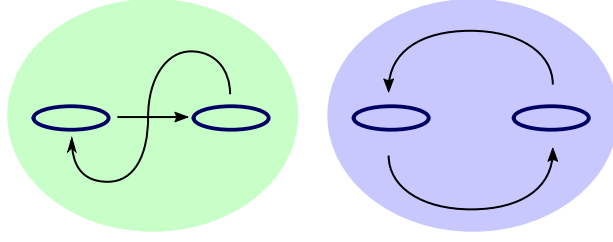
Figure 4.4 Disjoint-support representatives for the two generators of the abelianisation of the welded braid group $\mathbf{wB}_n$ for $n \geqslant 4$. The supports are shaded in green and blue respectively.

Proposition 4.12 *The residue $\Gamma_\infty(\mathbf{wB}_3)$ is normally generated by $\sigma_1\sigma_2^{-1}$ and $\tau_1\tau_2$. The quotient $G = \mathbf{wB}_3/\Gamma_\infty$ has the following presentation:*

$$G = \left\langle \sigma, \tau \mid \tau^2 = 1, \ \sigma^2\tau = \tau\sigma^2 \right\rangle.$$

Its center $\mathcal{Z}G$ is infinite cyclic, generated by σ^2, and:

$$G/\mathcal{Z}G \cong \mathbb{Z}/2 * \mathbb{Z}/2.$$

Proof. The canonical map $\mathbf{B}_3 \to \mathbf{wB}_3$ sends $\Gamma_\infty(\mathbf{B}_3)$ to $\Gamma_\infty(\mathbf{wB}_3)$. Since $\sigma_1\sigma_2^{-1} \in \Gamma_2(\mathbf{B}_3) = \Gamma_\infty(\mathbf{B}_3)$, we deduce that $\sigma_1\sigma_2^{-1} \in \Gamma_\infty(\mathbf{wB}_3)$. The same holds with $\tau_1\tau_2$, which is the image of $\tau_1\tau_2 \in \Gamma_2(\mathfrak{S}_3) = \Gamma_\infty(\mathfrak{S}_3)$ by the canonical map $\mathfrak{S}_3 \to \mathbf{wB}_3$. Thus, $\sigma_1\sigma_2^{-1}, \tau_1\tau_2 \in \Gamma_\infty(\mathbf{wB}_3)$, implying that $\mathbf{wB}_3/\Gamma_\infty$ is a quotient of the group G obtained from $\mathbf{wB}_3$ by adding the relations $\sigma_1 = \sigma_2$ and $\tau_1 = \tau_2$. We deduce from the usual presentation of $\mathbf{wB}_3$ that G is indeed described by the presentation of the statement. We are going to show that G is residually nilpotent, which implies that G is indeed equal to $\mathbf{wB}_3/\Gamma_\infty$ (it is the biggest residually nilpotent quotient of $\mathbf{wB}_3$). We first deduce from its presentation that the abelianisation of G is given by:

$$G^{\mathrm{ab}} \cong \mathbb{Z}\{\bar{\sigma}, \bar{\tau}\}/(2\bar{\tau}) \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

In particular, $\bar{\sigma}$ is not a torsion element of G^{ab}, which means that no non-trivial power of σ belongs to $\Gamma_2(G)$.

Now, consider the element $\sigma^2 \in G$. It commutes with σ and τ, thus it is central. In particular, $\langle\sigma^2\rangle$ is a normal subgroup. Moreover, the quotient $G/\langle\sigma^2\rangle$ is described by:

$$G/\langle\sigma^2\rangle \cong \left\langle \sigma, \tau \mid \sigma^2 = \tau^2 = 1, \ \sigma^2\tau = \tau\sigma^2 \right\rangle \cong \left\langle \sigma, \tau \mid \sigma^2 = \tau^2 = 1 \right\rangle,$$

which is exactly $\mathbb{Z}/2 * \mathbb{Z}/2$. Proposition B.16 then implies that G is residually nilpotent. Indeed, if $x \in \Gamma_\infty(G)$, then its image in $G/\langle\sigma^2\rangle \cong \mathbb{Z}/2 * \mathbb{Z}/2$ is in $\Gamma_\infty(\mathbb{Z}/2 * \mathbb{Z}/2) = \{1\}$. Thus $x \in \langle\sigma^2\rangle$. Since $\bar{\sigma}$ is not a torsion element in G^{ab}, we have $\langle\sigma^2\rangle \cap \Gamma_2 G = \{1\}$, whence $x = 1$.

Finally, we remark that $\mathbb{Z}/2 * \mathbb{Z}/2$ is centerless, implying that $\mathcal{Z}G$ consists only of $\langle\sigma^2\rangle$. As $\bar{\sigma}$ is not a torsion element in G^{ab}, $\langle\sigma^2\rangle$ has to be infinite cyclic, and all our claims are proved. $\square$

Remark 4.13 Both $\mathbf{vB}_3$ and $\mathbf{wB}_3$ have a non-stopping lower central series, without being residually nilpotent. In this situation, we can ask whether the transfinite lower central series is worth investigating: is our $\Gamma_\infty(G)$ (which is in fact $\Gamma_\omega(G)$) the intersection of all the terms of the transfinite lower central series of G, or are there further terms in the transfinite lower central series? In other words, is $[G, \Gamma_\omega(G)]$ equal $\Gamma_\omega(G)$, or is it smaller? In both cases, we have in fact $[G, \Gamma_\omega(G)] = \Gamma_\omega(G)$, so the transfinite lower central series stops at ω. Indeed, in both cases, $\Gamma_\omega(G)$ is normally generated by $\tau_1\tau_2^{-1}$ and $\sigma_1\sigma_2^{-1}$, belonging respectively to $\Gamma_2(\mathfrak{S}_3) = \Gamma_\omega(\mathfrak{S}_3)$ and $\Gamma_2(\mathbf{B}_3) = \Gamma_\omega(\mathbf{B}_3)$. Then:

$$\tau_1\tau_2^{-1} \in \Gamma_3(\mathfrak{S}_3) = [\mathfrak{S}_3, \Gamma_2(\mathfrak{S}_3)] = [\mathfrak{S}_3, \Gamma_\omega(\mathfrak{S}_3)] \subseteq [G, \Gamma_\omega(G)].$$

The same calculation implies that $[G, \Gamma_\omega(G)]$ contains $\sigma_1\sigma_2^{-1}$, so it must contain $\Gamma_\omega(G)$, as claimed. The same argument will lead to the same conclusion for $G = \widetilde{\mathbf{w}}\mathbf{B}_3$; see Proposition 4.16 and Remark 4.17.

4.4 Extended welded braids

Let us consider the space $F_{n\mathbb{S}^1}(\mathbb{D}^3)$ of configurations of n unlinked oriented circles in the unit 3-disc $\mathbb{D}^3$. By *oriented* circles, we mean that a configuration is defined up to *positive* re-parametrisation of the circles. The group $\mathfrak{S}_n$ acts on it by permuting the labels of the copies of $\mathbb{S}^1$. The group $(\mathbb{Z}/2)^n$ also acts on it, by changing the orientations of the different copies of $\mathbb{S}^1$. Both these actions combine into an action of the hyperoctahedral group $W_n = (\mathbb{Z}/2) \wr \mathfrak{S}_n$, which is obviously free. Taking quotients thus induces a commutative square of coverings:

$$\begin{array}{ccc} F_{n\mathbb{S}^1}(\mathbb{D}^3) & \longrightarrow\!\!\!\rightarrow & F_{n\mathbb{S}^1}(\mathbb{D}^3)/\mathfrak{S}_n \\ \downarrow & & \downarrow \\ F_{n\mathbb{S}^1}(\mathbb{D}^3)/(\mathbb{Z}/2)^n & \longrightarrow\!\!\!\rightarrow & F_{n\mathbb{S}^1}(\mathbb{D}^3)/W_n. \end{array} \qquad (4.1)$$

We recall that the fundamental groups of $F_{n\mathbb{S}^1}(\mathbb{D}^3)$ and $F_{n\mathbb{S}^1}(\mathbb{D}^3)/\mathfrak{S}_n$ identify respectively with $\mathbf{wP}_n$ and $\mathbf{wB}_n$. Precisely, the correspondence between loops in $F_{n\mathbb{S}^1}(\mathbb{D}^3)/\mathfrak{S}_n$ and the generators of $\mathbf{wB}_n$ is given as follows:

- τ_i corresponds to swapping the circles i and $i+1$ without either of them passing through the other;
- σ_i corresponds to swapping the circles i and $i+1$ while the $(i+1)$-th one goes through the i-th one in the direction determined by the orientation of the latter.

We denote the fundamental group of $F_{n\mathbb{S}^1}(\mathbb{D}^3)/W_n$ (resp. $F_{n\mathbb{S}^1}(\mathbb{D}^3)/(\mathbb{Z}/2)^n$) by $\widetilde{\mathbf{w}}\mathbf{B}_n$ (resp. $\widetilde{\mathbf{w}}\mathbf{P}_n$) and its elements are called *extended welded braids* (resp. *pure* extended welded braids). The vertical maps in the diagram (4.1) can be thought of as "forgetting the orientation of the circles". By taking fundamental groups of (4.1), we get a square of injections:

$$\begin{array}{ccc} \mathbf{wP}_n & \lhook\joinrel\longrightarrow & \mathbf{wB}_n \\ \downarrow & & \downarrow \\ \widetilde{\mathbf{w}}\mathbf{P}_n & \lhook\joinrel\longrightarrow & \widetilde{\mathbf{w}}\mathbf{B}_n. \end{array} \qquad (4.2)$$

Moreover, we know from the usual theory of covering spaces that the quotient $\widetilde{\mathbf{w}}\mathbf{B}_n/\mathbf{wP}_n$ is isomorphic to the acting group $(\mathbb{Z}/2) \wr \mathfrak{S}_n$. In fact, recalling that $\mathbb{Z}/2 \cong \pi_1(\mathbb{P}^2)$ we can give a geometric interpretation of the corresponding projection $\widetilde{\mathbf{w}}\mathbf{B}_n \twoheadrightarrow (\mathbb{Z}/2) \wr \mathfrak{S}_n$: a loop braid is sent to the collection of paths in $\mathbb{P}^2$ described by the unit vector normal to each circle, together with the permutation it induces.

Remark 4.14 We have been a little imprecise by what we mean by configurations of n unlinked, oriented circles. One natural meaning is the space of all embeddings of n disjoint copies of S^1 into the interior of $\mathbb{D}^3$ whose image is an unlink, equipped with the Whitney topology, quotiented by the action of orientation-preserving diffeomorphisms of the circles. Another natural meaning is the subspace of this space consisting of all such embedded unlinks where the embedded circles are *rigid*, in the sense that they are rotations, dilations and translations of the equator $S^1 \subset S^2 = \partial\mathbb{D}^3$. The latter space has the advantage of being a finite-dimensional manifold, and the above description involving the *unit normal vectors* associated to a configuration makes sense. We will freely move between these two viewpoints in light of the theorem of Brendle and Hatcher [BH13, Th. 4.1], which says that these two models are homotopy equivalent.

A presentation of the group $\widetilde{\mathbf{w}}\mathbf{B}_n$ was given as [BH13, Prop. 3.3 and 3.7]; see also Proposition A.3. It involves the presentation of $\mathbf{wB}_n$, to which are added generators ρ_i and several relations. As a loop braid, ρ_i corresponds to rotating the i-th circle by 180 degrees so that its orientation is reversed.

One can check from the presentation that the projection $\widetilde{\mathbf{w}}\mathbf{B}_n \twoheadrightarrow W_n = (\mathbb{Z}/2) \wr \mathfrak{S}_n$ identifies with the quotient of $\widetilde{\mathbf{w}}\mathbf{B}_n$ by the relations $\sigma_i = \tau_i$. This corresponds, on diagrams, to changing every crossing into a virtual one. Moreover, this projection splits, as one can see either from the geometry

or from the presentation: a basis of $(\mathbb{Z}/2)^n$ is sent to the generators ρ_i, and $\mathfrak{S}_n$ identifies with the subgroup of $\mathbf{wB}_n$ generated by the τ_i. Thus we get the decomposition:

$$\widetilde{\mathbf{w}}\mathbf{B}_n \cong \mathbf{wP}_n \rtimes W_n.$$

Lemma 4.15 *For $n \geqslant 2$, the abelianisation $\widetilde{\mathbf{w}}\mathbf{B}_n^{\mathrm{ab}}$ is isomorphic to $(\mathbb{Z}/2)^3$. A basis over $\mathbb{Z}/2$ is given by σ, τ and ρ, which are respectively the common class of the σ_i, the τ_i and the ρ_i.*

Proof. On can get this directly from the presentation of Proposition A.3. One can also see this as a particular case of Proposition 4.21 below. $\square$

Proposition 4.16 *The lower central series of the extended welded braid group $\widetilde{\mathbf{w}}\mathbf{B}_n$:*

- stops at Γ_2 *if $n \geqslant 4$.*
- does not stop *if $n = 2, 3$.*

Proof. **Case 1:** $n \geqslant 4$**.** As in the proof of Proposition 4.9, we show that each pair of generators of $\widetilde{\mathbf{w}}\mathbf{B}_n^{\mathrm{ab}}$ may be represented by elements of $\widetilde{\mathbf{w}}\mathbf{B}_n$ with disjoint support. For the first two generators in the list, this is possible since each of them may be supported inside a 3-ball in $\mathbb{D}^3$ containing two of the circles; see Figure 4.4. For one of the first two generators together with the third generator, this is possible even for $n \geqslant 3$, since the third generator may be supported in a 3-ball containing a single circle; see Figure 4.5.

Case 2: $n = 2$**.** For short, we denote σ_1 (resp. τ_1) by σ (resp. τ) From the presentation of $\widetilde{\mathbf{w}}\mathbf{B}_2$, we can see that the quotient by the relations $\rho_1 = \rho_2 = 1$ admits the presentation:

$$\widetilde{\mathbf{w}}\mathbf{B}_2/(\rho_1, \rho_2) = \langle \sigma, \tau \mid \tau^2 = 1, \ \sigma = \tau\sigma^{-1}\tau \rangle.$$

But this is a presentation of $\mathbb{Z} \rtimes (\mathbb{Z}/2)$, whose lower central series, computed in Corollary B.8, does not stop.

Case 3: $n = 3$**.** It is known that $\widetilde{\mathbf{w}}\mathbf{B}_3$ contains $\mathbf{B}_3$ as a subgroup, hence it cannot be residually nilpotent. In fact, $\Gamma_\infty(\widetilde{\mathbf{w}}\mathbf{B}_3)$ contains $\Gamma_\infty(\mathfrak{S}_3) = \Gamma_2(\mathfrak{S}_3)$ and $\Gamma_\infty(\mathbf{B}_3) = \Gamma_2(\mathbf{B}_3)$, which contain respectively $\tau_2\tau_1$ and $\sigma_2\sigma_1^{-1}$. Let us consider the quotient G of $\widetilde{\mathbf{w}}\mathbf{B}_3$ by these two relations, and let us denote by τ (resp. σ) the common image of τ_1 and τ_2 (resp. σ_1 and σ_2) in G. We remark that in G, $\rho_2 = \tau_1\rho_1\tau_1$ and $\rho_3 = \tau_2\rho_2\tau_2$ are both identified with $\rho_1 = \tau_2\rho_1\tau_2 = \tau_1\rho_2\tau_1$, so we can also speak of their common class ρ. From the presentation of $\widetilde{\mathbf{w}}\mathbf{B}_3$, we get a presentation of G:

$$G = \langle \sigma, \tau, \rho \mid \rho^2 = \tau^2 = 1; \ \sigma, \tau \rightleftarrows \rho; \ \sigma\rho = \rho\tau\sigma^{-1}\tau \rangle.$$

The second group of relations, saying ρ is central, may be seen as *disjoint support* relations. Since ρ is central, the last relation is equivalent to $\sigma = \tau\sigma^{-1}\tau$. As a consequence, this presentation is a presentation of $\mathbb{Z}/2 \times (\mathbb{Z} \rtimes (\mathbb{Z}/2))$, whose lower central series does not stop, again thanks to Corollary B.8. $\square$

Remark 4.17 In the third part of the proof, the fact that the quotient G is residually nilpotent implies that it is the quotient by the whole of $\Gamma_\infty(\widetilde{\mathbf{w}}\mathbf{B}_3)$, which is thus normally generated by $\tau_2\tau_1$ and $\sigma_2\sigma_1^{-1}$. This also means that $\mathcal{L}(\widetilde{\mathbf{w}}\mathbf{B}_3) \cong \mathcal{L}(G)$, which is easy to compute completely; see Corollary B.8.

4.5 Partitioned virtual, welded and extended welded braids

We now turn our attention to partitioned virtual and welded braids, which are defined exactly in the same way as partitioned braids have been in Definition 3.2.
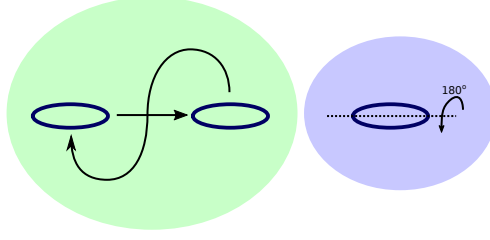
Figure 4.5 Disjoint-support representatives for two of the generators of the abelianisation of the extended welded braid group $\widetilde{\mathbf{w}}\mathbf{B}_n$ for $n \geqslant 3$. The supports are shaded in green and blue respectively.

4.5.1 Basic theory

Let us consider the groups $\mathbf{vB}_n$, $\mathbf{wB}_n$ and $\widetilde{\mathbf{w}}\mathbf{B}_n$. Each one is endowed with a canonical projection π onto $\mathfrak{S}_n$ (we use the same notation in the three cases). One can think of it as the replacement of every crossing in a diagram by virtual ones (and forgetting the mark points for extended welded braids); in terms of group presentations, it sends both σ_i and τ_i to the transposition $(i, i+1)$, for each i (and it kills the ρ_i).

Definition 4.18 Let $n \geqslant 1$ be an integer, and let $\lambda = (n_1, ..., n_l)$ be a partition of n. The corresponding *partitioned virtual braid group* is:

$$\mathbf{vB}_\lambda := \pi^{-1}(\mathfrak{S}_\lambda) = \pi^{-1}\left(\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}\right) \subseteq \mathbf{vB}_n.$$

We define similarly the *partitioned welded braid group* $\mathbf{wB}_\lambda := \pi^{-1}(\mathfrak{S}_\lambda)$ and the group of λ-*partitioned extended welded braid group* $\widetilde{\mathbf{w}}\mathbf{B}_\lambda = \pi^{-1}(\mathfrak{S}_\lambda)$.

When working with virtual (or welded) braids, some things are made easier by the fact that the canonical projection onto $\mathfrak{S}_n$ splits, so that $\mathbf{vB}_n \cong \mathbf{vP}_n \rtimes \mathfrak{S}_n$ (resp. $\mathbf{wB}_n \cong \mathbf{wP}_n \rtimes \mathfrak{S}_n$). This works for partitioned braids too, since this splitting restricts obviously to a splitting of $\mathbf{vB}_\lambda \twoheadrightarrow \mathfrak{S}_\lambda$ (resp. of $\mathbf{wB}_\lambda \twoheadrightarrow \mathfrak{S}_\lambda$), so that:

$$\mathbf{vB}_\lambda \cong \mathbf{vP}_n \rtimes \mathfrak{S}_\lambda, \tag{4.3}$$

and likewise for welded braids. Extended welded braids behave much in the same way: the decomposition $\widetilde{\mathbf{w}}\mathbf{B}_n \cong \mathbf{wP}_n \rtimes W_n$ restricts to a decomposition:

$$\widetilde{\mathbf{w}}\mathbf{B}_\lambda \cong \mathbf{wP}_n \rtimes W_\lambda, \tag{4.4}$$

where W_λ denotes the subgroup $(\mathbb{Z}/2) \wr \mathfrak{S}_\lambda$ of $W_n = (\mathbb{Z}/2) \wr \mathfrak{S}_n$.

Our analysis of $\mathbf{vB}_n$, $\mathbf{wB}_n$ and $\widetilde{\mathbf{w}}\mathbf{B}_n$ will rely on the following classical results (see also §A.1.3):

Proposition 4.19 *The group $\mathbf{vP}_n$ (resp. $\mathbf{wP}_n$) is generated by the elements χ_{ij} $(1 \leqslant i \neq j \leqslant n)$ depicted in Figure 4.6 and $\mathbf{vP}_n^{\mathrm{ab}}$ (resp. $\mathbf{wP}_n^{\mathrm{ab}}$) is free on the classes of the χ_{ij}.*

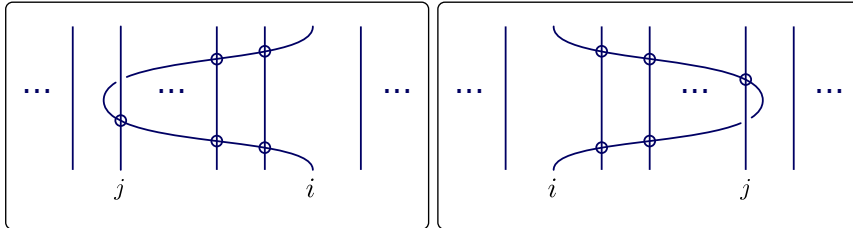

Figure 4.6 The standard generators χ_{ij} of pure virtual and welded braid groups.

The proof of Lemma 3.3 can be adapted easily to the present context, giving:

Lemma 4.20 *Let $\lambda = (n_1, ..., n_l)$ be a partition of an integer $n \geqslant 1$. Then $\mathbf{vB}_\lambda$ (resp. $\mathbf{wB}_\lambda$) is the subgroup of $\mathbf{vB}_n$ (resp. $\mathbf{wB}_n$) generated by:*

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- *The σ_α and the τ_α for $1 \leqslant \alpha \leqslant n$ such that α and $\alpha + 1$ are in the same block of λ.*
- *The $\chi_{\alpha\beta}$, for $1 \leqslant \alpha \neq \beta \leqslant n$ such that α and β do not belong to the same block of λ.*

Also, $\widetilde{\mathbf{w}}\mathbf{B}_\lambda$ is the subgroup of $\widetilde{\mathbf{w}}\mathbf{B}_n$ generated by the same set of generators, together with the ρ_α for all $\alpha \leqslant n$.

We can use this to compute abelianisations:

Proposition 4.21 *Let $\lambda = (n_1, ..., n_l)$ be a partition of an integer $n \geqslant 1$, and let l' be the number of i such that $n_i \geqslant 2$. Then:*

$$\mathbf{v}\mathbf{B}_\lambda^{\mathrm{ab}} \cong \mathbf{w}\mathbf{B}_\lambda^{\mathrm{ab}} \cong (\mathbb{Z}/2)^{l'} \times \mathbb{Z}^{l'} \times \mathbb{Z}^{l(l-1)} \quad and \quad \widetilde{\mathbf{w}}\mathbf{B}_\lambda^{\mathrm{ab}} \cong (\mathbb{Z}/2)^{l+2l'+l(l-1)}.$$

Precisely, the abelian group $\mathbf{v}\mathbf{B}_\lambda^{\mathrm{ab}}$ ($\cong \mathbf{w}\mathbf{B}_\lambda^{\mathrm{ab}}$) is generated by:

- *For each $i \in \{1, ..., l\}$ such that $n_i \geqslant 2$, one generator s_i (resp. t_i): this is the common class of the σ_α (resp. the τ_α) for α and $\alpha + 1$ in the i-th block of λ.*
- *For each $i, j \in \{1, ..., l\}$ such that $i \neq j$, one generator x_{ij}: this is the common class of the $\chi_{\alpha\beta}$ for α in the i-th block of λ and β in the j-th one.*

The t_i are a $\mathbb{Z}/2$-basis of the first factor, the s_i are a $\mathbb{Z}$-basis of the second one, and the x_{ij} are a $\mathbb{Z}$-basis of the third one.

A $(\mathbb{Z}/2)$-basis of $\widetilde{\mathbf{w}}\mathbf{B}_\lambda^{\mathrm{ab}} \cong (\mathbf{w}\mathbf{B}_\lambda^{\mathrm{ab}} \otimes \mathbb{Z}/2) \times (\mathbb{Z}/2)^l$ is given by the same list of generators (which are of 2-torsion in $\widetilde{\mathbf{w}}\mathbf{B}_\lambda^{\mathrm{ab}}$), together with:

- *For each $i \in \{1, ..., l\}$, one generator r_i: this is the common class of the ρ_α for α in the i-th block of λ.*

Proof. We show our first statement for virtual braids; the exact same proof works for $\mathbf{w}\mathbf{B}_\lambda$.

Applying Lemma 1.9 to the decomposition (4.3), we deduce that $\mathbf{v}\mathbf{B}_\lambda^{\mathrm{ab}} \cong (\mathbf{v}\mathbf{P}_n^{\mathrm{ab}})_{\mathfrak{S}_\lambda} \times \mathfrak{S}_\lambda^{\mathrm{ab}}$. On the one hand, $\mathfrak{S}_{n_i}^{\mathrm{ab}} \cong \mathbb{Z}/2$ (generated by the class of any transposition) for all i such that $n_i \geqslant 2$, so that $\mathfrak{S}_\lambda^{\mathrm{ab}} \cong (\mathbb{Z}/2)^l$ admits the t_i as a $\mathbb{Z}/2$-basis. On the other hand, $\mathbf{v}\mathbf{P}_n^{\mathrm{ab}}$ is freely generated by the set X of the $\overline{\chi}_{\alpha\beta}$ with $1 \leqslant \alpha \neq \beta \leqslant n$, on which $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_n$ acts by permuting the indices. Thus $(\mathbf{v}\mathbf{P}_n^{\mathrm{ab}})_{\mathfrak{S}_\lambda}$ is free on the set $X/\mathfrak{S}_\lambda$. The latter is exactly the set of generators x_{ij} described in the statement, plus one additional element for each i such that $n_i \geqslant 2$, which is the common class x_i of the $\chi_{\alpha\beta}$ for α and β in the i-th block. Thus we get the decomposition of the statement, with a slightly different basis (the x_i instead of the s_i). The relation $\chi_{\alpha,\alpha+1} = \tau_\alpha\sigma_\alpha$ implies that $s_i = x_i - t_i$, which is the change of basis we need to finish proving our first statement.

Let us now describe $\widetilde{\mathbf{w}}\mathbf{B}_\lambda^{\mathrm{ab}}$, using a similar computation. We deduce from the decomposition (4.4) that $\widetilde{\mathbf{w}}\mathbf{B}_\lambda^{\mathrm{ab}} \cong (\mathbf{w}\mathbf{P}_n^{\mathrm{ab}})_{W_\lambda} \times W_\lambda^{\mathrm{ab}}$. On the one hand, the group $W_\lambda = (\mathbb{Z}/2) \wr \mathfrak{S}_\lambda$ is isomorphic to the product of the $(\mathbb{Z}/2) \wr \mathfrak{S}_{n_i}$, whose abelianisation is $\mathbb{Z}/2$ (generated by r_i) if $n_i = 1$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$ (generated by r_i and s_i) if $n_i \geqslant 2$. Thus $W_\lambda^{\mathrm{ab}} \cong (\mathbb{Z}/2)^{l+l'}$ gives a first factor in our formula. On the other hand the action of the generator ρ_γ of $(\mathbb{Z}/2)^n \subset W_\lambda$ on $\mathbf{w}\mathbf{P}_n^{\mathrm{ab}}$ is by fixing all the $\overline{\chi}_{\alpha\beta}$ for $\beta \neq \gamma$ and sending all $\overline{\chi}_{\alpha\gamma}$ to $-\overline{\chi}_{\alpha\gamma}$. One can check this from the presentation of Proposition A.9: for instance, $\chi_{12} = \tau_1\sigma_1$, and $\rho_1\chi_{12}\rho_1^{-1} = \sigma_1^{-1}\tau_1 = \chi_{12}^{-1}$. As a consequence, $(\mathbf{v}\mathbf{P}_n^{\mathrm{ab}})_{(\mathbb{Z}/2)^n} \cong \mathbf{v}\mathbf{P}_n^{\mathrm{ab}} \otimes \mathbb{Z}/2$ is a free $\mathbb{Z}/2$-vector space on the $\overline{\chi}_{\alpha\beta}$, on which $\mathfrak{S}_n$ acts by permutation of the indices. Since $(\mathbf{w}\mathbf{P}_n^{\mathrm{ab}})_{W_\lambda} = (\mathbf{w}\mathbf{P}_n^{\mathrm{ab}})_{(\mathbb{Z}/2) \wr \mathfrak{S}_\lambda} = ((\mathbf{v}\mathbf{P}_n^{\mathrm{ab}})_{(\mathbb{Z}/2)^n})_{\mathfrak{S}_\lambda}$, the rest of the proof works exactly as for the previous case, except that we work with coefficients in $\mathbb{Z}/2$. $\square$

4.5.2 The lower central series

The remainder of the section is devoted to the proof of the following:

Theorem 4.22 *Let $n \geqslant 1$ be an integer, and let $\lambda = (n_1, ..., n_l)$ be a partition of n. The lower central series of the partitioned virtual (resp. welded) braid group $\mathbf{v}\mathbf{B}_\lambda$ (resp. $\mathbf{w}\mathbf{B}_\lambda$):*

- *stops at Γ_2 if $n_i \geqslant 4$ for all i, or if $n_j = 1$ for exactly one j and $n_i \geqslant 4$ for all $i \neq j$.*

- does not stop *in all the other cases.*

The lower central series of the partitioned extended welded braid group $\widetilde{\mathbf{w}}\mathbf{B}_\lambda$:

- stops at Γ_2 *if* $n_i \geqslant 4$ *for all* i.
- does not stop *in all the other cases, except if* $\lambda = (1)$, *for* $\widetilde{\mathbf{w}}\mathbf{B}_1 \cong \mathbb{Z}/2$.

Proof. First suppose that $n_i \geqslant 4$ for all i. As in the proofs of Propositions 4.9 and 4.11, we show that any pair of generators of $\mathbf{v}\mathbf{B}_\lambda^{\mathrm{ab}}$ (resp. of $\mathbf{w}\mathbf{B}_\lambda^{\mathrm{ab}}$, resp. of $\widetilde{\mathbf{w}}\mathbf{B}_\lambda^{\mathrm{ab}}$) may be represented by a pair of elements of the group having *disjoint support* (in the sense of Definition 4.4 for virtual braids, and of the equivalent Definitions 4.1 and 4.4 for welded braids), which therefore commute by Facts 4.3 and 4.7. The generators we need to consider are listed in Proposition 4.21: they are the s_i, the t_i, the x_{ij} and (in the case of extended welded braids) the r_i. In each case, it is clear that the hypothesis $n_i \geqslant 4$ allows us to choose lifts with disjoint support among the generators of $\mathbf{v}\mathbf{B}_\lambda$ (resp. of $\mathbf{w}\mathbf{B}_\lambda$, resp. of $\widetilde{\mathbf{w}}\mathbf{B}_\lambda$) from Lemma 4.20. Indeed, the class of any such generator depends only on the blocks to which the indices involved belong, and the support of each generator contains at most two indices. For instance, if α denotes the minimum of the i-th block, s_i and t_i can be lifted respectively to σ_α (whose support is $\{\alpha, \alpha+1\}$) and to $\tau_{\alpha+2}$ (whose support is $\{\alpha+2, \alpha+3\}$).

If λ has one block of size 2 or 3, then there is a surjection (defined by forgetting all strands except for those corresponding to this block) from $\mathbf{v}\mathbf{B}_\lambda$ onto $\mathbf{v}\mathbf{B}_2$ or $\mathbf{v}\mathbf{B}_3$, thus we can use Lemma 1.1 to deduce from Proposition 4.9 that its lower central series does not stop. The same argument applies to welded braids (resp. to extended welded braids), using Proposition 4.11 (resp. Proposition 4.16) instead of Proposition 4.9.

Similarly, if there are two blocks of size 1, then both $\mathbf{v}\mathbf{B}_\lambda$ and $\mathbf{w}\mathbf{B}_\lambda$ surject onto $\mathbf{v}\mathbf{B}_{1,1} = \mathbf{w}\mathbf{B}_{1,1} = \mathbf{v}\mathbf{P}_2 \cong \mathbf{F}_2$, whose lower central series does not stop. Under this hypothesis, $\widetilde{\mathbf{w}}\mathbf{B}_\lambda$ surjects onto $\widetilde{\mathbf{w}}\mathbf{B}_{1,1} \cong \mathbf{F}_2 \rtimes (\mathbb{Z}/2)^2$, whose lower central series does not stop either; see Remark 4.29.

Finally, we treat separately the tricky cases when there is exactly one block of size 1, all other blocks having size at least 4. For virtual and welded braids, we show in Proposition 4.26 and Proposition 4.24 that the lower central series stops at Γ_2. For extended welded braids, the fact that it does not stop is Proposition 4.28. $\qquad\square$

Remark 4.23 (Blocks of size 3) If λ has blocks of size 3 (say, $n_1 = 3$), the lower central series of G_λ (with $G = \mathbf{v}\mathbf{B}$, $\mathbf{w}\mathbf{B}$ or $\widetilde{\mathbf{w}}\mathbf{B}$) does not stop: the disjoint support argument fails, since t_1 and s_1 do not have lifts with disjoint support. However, this failure is limited, and the argument can still be applied to understand the lower central series of $G_{3,\mu}$ from the one of G_μ (where μ is any partition). Precisely, the generators t_1 and s_1 of the Lie ring do not commute with each other, but they do commute with every other generator. Using the canonical split injection of G_3 into $G_{3,\mu}$, we see that t_1 and s_1 (together with r_1 for extended welded braids) generate a copy of $\mathcal{L}(G_3)$, which is then a direct factor of $\mathcal{L}(G_{3,\mu})$. If all the blocks of μ have size at least 3, we can be even more precise: then the $2(l-1)$ elements x_{1i} and x_{i1} must be central. In this case, we get an isomorphism of Lie rings:

$$\mathcal{L}(G_{3,\mu}) \cong \mathcal{L}(G_3) \times \mathbb{Z}^{2(l-1)} \times \mathcal{L}(G_\mu).$$

Moreover, $\mathcal{L}(G_3)$ is well-understood (see Propositions 4.10 and 4.12, and the proof of Proposition 4.16), so this allows us to compute $\mathcal{L}(G_\lambda)$ completely if all the blocks of λ have size at least 3.

Proposition 4.24 *Let μ be a partition of an integer n into blocks of size at least* 4. *Then the lower central series of* $\mathbf{w}\mathbf{B}_{1,\mu}$ *stops at* Γ_2.

Proof. The proof is similar to that of Proposition 3.8, the key input being Lemma 4.25, which is the analogue of Lemma 3.10 in the classical case.

Let us denote by $\lambda = (1, n_2, ..., n_l)$ the partition $(1, \mu)$ of $n+1$. The canonical morphism $\mathbf{w}\mathbf{B}_\mu \hookrightarrow \mathbf{w}\mathbf{B}_{1,\mu}$ induces a morphism:

$$\mathbf{w}\mathbf{B}_\mu^{\mathrm{ab}} = \mathbf{w}\mathbf{B}_\mu/\Gamma_\infty \to \mathbf{w}\mathbf{B}_{1,\mu}/\Gamma_\infty,$$

where the equality on the left comes from the fact that all the blocks of μ have size at least 4. This implies that among the generators of $\mathbf{wB}_{1,\mu}$ from Proposition 4.20, the ones coming from $\mathbf{wB}_\mu$ pairwise commute. Moreover, this also implies that $\chi_{\alpha\beta}$ and $\chi_{\alpha'\beta'}$ have the same image x_{ij} in $\mathbf{wB}_{1,\mu}/\Gamma_\infty$ if α and α' are in the i-th block of λ and β and β' are in the j-th one, if $i,j \geqslant 2$. Similarly, σ_α and $\sigma_{\alpha'}$ (resp. τ_α and $\tau_{\alpha'}$) have the same image s_i (resp. t_i) in this quotient if α and α' are in the i-th block of λ, with $i \geqslant 2$. Finally, a generator of the form $\chi_{1\alpha}$ (resp. $\chi_{\alpha 1}$) is also sent to a class x_{1i} (resp. x_{i1}) modulo Γ_∞ depending only on the index $i \geqslant 2$ of the block containing α. Indeed, let us suppose that α and $\alpha + 1$ are in the i-th block of λ. This block has size at least 4, so t_i has a representative τ_β in $\mathbf{wB}_{1,\mu}$ whose support does not contain α, which then commutes with $\chi_{1\alpha}$. But t_i is also the class of τ_α, and $\tau_\alpha \chi_{1\alpha} \tau_\alpha = \chi_{1,\alpha+1}$. Thus $\overline{\chi}_{1\alpha} = t_i \overline{\chi}_{1\alpha} t_i = \overline{\chi}_{1,\alpha+1}$, and the same argument shows that $\overline{\chi}_{\alpha 1} = \overline{\chi}_{\alpha+1,1}$.

In order to prove our statement, we need to show that the x_{1i} and the x_{j1} commute with each other, and with all the other generators of $\mathbf{wB}_{1,\mu}/\Gamma_\infty$. The latter assertion is deduced from the fact that the pairs of generators involved have lifts to $\mathbf{wB}_{1,\mu}$ with disjoint support. The former one is a bit trickier, and requires the use of Lemma 4.25 below. Precisely, let $i,j \geqslant 2$ with $i \neq j$.

- x_{i1} and x_{j1} always commute, since $\chi_{\alpha 1}$ commutes with $\chi_{\beta 1}$, for every α, β.
- We have seen that x_{1i} commutes with x_{ij}. Thus, if α is in the i-th block and β in the j-th block of λ, the morphism $\langle \chi_{12}, \chi_{23}, \chi_{13} \rangle \to \mathbf{wB}_{1,\mu}/\Gamma_\infty$ induced by $1 \mapsto 1$, $2 \mapsto \alpha$ and $3 \mapsto \beta$ factors through the quotient by $[\chi_{12}, \chi_{23}]$, which is abelian, by Lemma 4.25. Thus x_{1i} commutes with x_{1j}.
- x_{i1} and x_{1j} are also in a homomorphic image of $\langle \chi_{12}, \chi_{13}, \chi_{23} \rangle$, where the image x_{i1} of χ_{12} commutes with the image x_{ij} of χ_{13}. Thus, again by Lemma 4.25, they commute.
- Finally, x_{i1} and x_{1i} are the classes of $\chi_{\alpha 1}$ and $\chi_{1,\alpha+1}$ respectively, with α and $\alpha + 1$ both in the i-th block of λ. Again, these are in a homomorphic image of $\langle \chi_{12}, \chi_{23}, \chi_{13} \rangle$, where the image x_{1i} of χ_{12} commutes with the image $\overline{\chi}_{\alpha,\alpha+1} = \overline{\tau_\alpha \sigma_\alpha} = t_i s_i$ of χ_{13}. Thus x_{i1} commutes with x_{1i}.

Finally, we have shown that all the generators of $\mathbf{wB}_{1,\mu}/\Gamma_\infty$ commute with each other, thus it is abelian, and $\Gamma_\infty = \Gamma_2$ for $\mathbf{wB}_{1,\mu}$. $\qquad\square$

Lemma 4.25 *Let G be the subgroup $\langle \chi_{12}, \chi_{23}, \chi_{13} \rangle$ of $\mathrm{Aut}(\mathbf{F}_n)$. The subgroup $\Gamma_2(G)$ is normally generated by $[\chi_{12}, \chi_{23}]$ (resp. by $[\chi_{12}, \chi_{13}]$). In other words, the quotient of G by the single relation $[\chi_{12}, \chi_{23}] = 1$ (resp. $[\chi_{12}, \chi_{13}] = 1$) is abelian.*

Proof. We know that χ_{13} already commutes with χ_{23} in G. Moreover, χ_{12} commutes with $\chi_{13}\chi_{23}$. Thus, χ_{12} and χ_{23} commute in a quotient of G if and only if χ_{12} commutes with $\chi_{13} = (\chi_{13}\chi_{23})\chi_{23}^{-1}$, and then the quotient is abelian. $\qquad\square$

The above proof can be adapted to work for *virtual* braids too, with an appropriate modification of Lemma 4.25, given by Lemma 4.27 below.

Proposition 4.26 *Let μ be a partition of an integer n into blocks of size at least 4. Then the lower central series of $\mathbf{vB}_{1,\mu}$ stops at Γ_2.*

Proof. We use the same notations as in the proof of Proposition 4.24. The first part of that proof adapts verbatim to the setting of virtual braids. First, the canonical homomorphism

$$\mathbf{vB}_\mu^{\mathrm{ab}} = \mathbf{vB}_\mu/\Gamma_\infty \longrightarrow \mathbf{vB}_{1,\mu}/\Gamma_\infty$$

implies that those generators of $\mathbf{vB}_{1,\mu}$ that come from $\mathbf{vB}_\mu$ pairwise commute in the quotient $\mathbf{vB}_{1,\mu}/\Gamma_\infty$. It also implies, just as above, that the images in $\mathbf{vB}_{1,\mu}/\Gamma_\infty$ of the generators of $\mathbf{vB}_{1,\mu}$ from Lemma 4.20 depend only on the *blocks* to which α and β belong. We therefore just have to show that the generators x_{1i} and x_{j1} (for $i,j \geqslant 2$) commute with each other, and with the other generators of $\mathbf{vB}_{1,\mu}/\Gamma_\infty$. The latter follows from the same disjoint support argument as in the proof of Proposition 4.24, so we are left with showing that they commute with each other.

There are four cases to consider, as in the proof of Proposition 4.24, and they may all be treated analogously. Let G denote the subgroup of the pure virtual braid group $\mathbf{vP}_3$ generated by the

elements χ_{12}, χ_{13} and χ_{23}. In each case, we define a homomorphism $\theta : G \to \mathbf{vB}_{1,\mu}/\Gamma_\infty$ such that two of the elements χ_{12}, χ_{13} and χ_{23} are sent to the elements of $\mathbf{vB}_{1,\mu}/\Gamma_\infty$ that we wish to show commute, and the third is sent to an element that commutes with both of them. It then follows from Lemma 4.27 below that the first two elements commute. In detail, the four cases are as follows, for $i, j \geqslant 2$ with $i \neq j$:

- We wish to show that x_{i1} and x_{j1} commute. Let θ be induced by $1 \mapsto \alpha$, $2 \mapsto \beta$ and $3 \mapsto 1$, with α in the i-th block and β in the j-th block of λ. The image of χ_{12} is x_{ij}, which commutes with $x_{i1} = \theta(\chi_{13})$ and $x_{j1} = \theta(\chi_{23})$, as we have deduced above from a disjoint support argument.
- The case of x_{1j} and x_{1i} is similar to the previous one, the roles of 1 and 3 being exchanged.
- Both x_{i1} and x_{1j} commute with x_{ij}, and again the same argument implies that they commute.
- Finally, x_{i1} and x_{1i} are the classes of $\chi_{\alpha 1}$ and $\chi_{1,\alpha+1}$ respectively, with α and $\alpha + 1$ both in the i-th block of λ. Since x_{i1} and x_{1i} both commute with $\overline{\chi}_{\alpha,\alpha+1} = t_i s_i$, we can apply once again the same argument, θ being induced by $1 \mapsto 1$, $2 \mapsto \alpha$ and $3 \mapsto \alpha + 1$.

We have therefore shown that all generators in a generating set for $\mathbf{vB}_{1,\mu}/\Gamma_\infty$ pairwise commute. Hence this quotient is abelian, in other words, $\Gamma_\infty = \Gamma_2$ for $\mathbf{vB}_{1,\mu}$. $\qquad\square$

Lemma 4.27 *Let G be the subgroup of $\mathbf{vP}_3$ generated by the elements χ_{12}, χ_{13} and χ_{23}. Then any two of the three relations*

$$[\chi_{12}, \chi_{23}] = 1 \qquad [\chi_{13}, \chi_{23}] = 1 \qquad [\chi_{12}, \chi_{13}] = 1$$

imply the third. In other words, the quotient of G by any two of these relations is abelian.

Proof. The elements χ_{12}, χ_{13} and χ_{23} satisfy the relation

$$\chi_{12}\chi_{13}\chi_{23}\chi_{12}^{-1}\chi_{13}^{-1}\chi_{23}^{-1} = 1 \tag{4.5}$$

in $\mathbf{vP}_3$; see [Bar04, Th. 1]. If we impose any two of the three relations above, then one of the three generators becomes central, and (4.5) becomes the relation stating that the other two generators commute. $\qquad\square$

Proposition 4.28 *For any partition μ of an integer $n \geqslant 1$, the lower central series of $\widetilde{\mathbf{w}}\mathbf{B}_{1,\mu}$ does not stop.*

Proof. Forgetting every block of μ save one induces a surjection from $\widetilde{\mathbf{w}}\mathbf{B}_{1,\mu}$ onto $\widetilde{\mathbf{w}}\mathbf{B}_{1,m}$ for some $m \geqslant 1$, so it is enough to show that the lower central series of $\widetilde{\mathbf{w}}\mathbf{B}_{1,m}$ does not stop by Lemma 1.1. Let us consider $\widetilde{\mathbf{w}}\mathbf{B}_{1,m}$ as a subgroup of $\mathrm{Aut}(\mathbf{F}_{1+m})$. One can check on the generators of $\widetilde{\mathbf{w}}\mathbf{B}_{1,m}$ that the automorphisms in $\widetilde{\mathbf{w}}\mathbf{B}_{1,m}$ preserve the normal subgroup of $\mathbf{F}_{1+m}$ generated by the $x_\alpha x_{\alpha+1}^{-1}$ for $2 \leqslant \alpha \leqslant m$, together with the x_α^2 for $2 \leqslant \alpha \leqslant m + 1$. As a consequence, the quotient of $\mathbf{F}_{1+m} \cong \mathbb{Z} * \mathbf{F}_m$ onto $\mathbb{Z} * \mathbb{Z}/2$ induces a well-defined morphism $\widetilde{\mathbf{w}}\mathbf{B}_{1,m} \to \mathrm{Aut}(\mathbb{Z} * \mathbb{Z}/2)$. Let us denote again by x_1 and x_2 the generators of $\mathbb{Z} * \mathbb{Z}/2$, which are the images of $x_1, x_2 \in \mathbf{F}_{1+m}$. The image G of $\widetilde{\mathbf{w}}\mathbf{B}_{1,m}$ in $\mathrm{Aut}(\mathbb{Z} * \mathbb{Z}/2)$ is generated by ρ_1, χ_{12} and χ_{21}, which are defined by the same formulas as the corresponding elements of $\mathrm{Aut}(\mathbf{F}_2)$, only with x_2 having square 1. We note that χ_{21} and χ_{12} are conjugation by x_1 and by x_2 on $\mathbb{Z} * \mathbb{Z}/2$ respectively, so they generate $\mathrm{Inn}(\mathbb{Z} * \mathbb{Z}/2) \cong \mathbb{Z} * \mathbb{Z}/2$, which is normal in $\mathrm{Aut}(\mathbb{Z} * \mathbb{Z}/2)$, whence also in G. Moreover, ρ_1 is not an inner automorphism and, since $\rho_1^2 = 1$, we have:

$$G = \langle \chi_{21}, \chi_{12} \rangle \rtimes \langle \rho_1 \rangle \cong (\mathbb{Z} * \mathbb{Z}/2) \rtimes (\mathbb{Z}/2).$$

Conjugation by ρ_1 fixes χ_{12} and sends χ_{21} to its inverse. Hence the quotient of G by χ_{12} is isomorphic to $\mathbb{Z} \rtimes (\mathbb{Z}/2)$, whose lower central series does not stop by Proposition B.16. $\qquad\square$

Remark 4.29 If $m = 1$, then the same kind of calculation can be applied directly in $\mathrm{Aut}(\mathbf{F}_2)$ to see that $\widetilde{\mathbf{w}}\mathbf{B}_{1,1} = \widetilde{\mathbf{w}}\mathbf{P}_2 \cong \mathbf{F}_2 \rtimes (\mathbb{Z}/2)^2$, where each generator of $(\mathbb{Z}/2)^2$ fixes one of the generators of $\mathbf{F}_2$, and sends the other one to its inverse. This group surjects onto $(\mathbb{Z} \rtimes (\mathbb{Z}/2)) \times (\mathbb{Z}/2)$ (by killing one of the generators of $\mathbf{F}_2$), so its lower central series does not stop.

5 Variants on partitioned welded braids

Here we study our most general version of a group of partitioned welded braids, built from configurations of points, oriented circles and unoriented ones.

5.1 Pure bipartite welded braids

We begin with introducing the pure version of this group, obtained from configurations of points and oriented circles. Namely, let k and m be integers. The *pure bipartite* welded braid group $\mathbf{wP}(k,m)$ is defined as the fundamental group of the configuration space $F_{k \sqcup m \mathbb{S}^1}(\mathbb{D}^3)$. In order to study it, we identify this group with a subgroup of $\mathbf{wP}_{k+m}$. This is done by considering a map between configuration spaces. Precisely, we consider the map from the above configuration space to $F_{(k+m)\mathbb{S}^1}(\mathbb{D}^3)$ obtained by replacing each point in a configuration by a small (oriented) circle. This can be done explicitly: one can replace each point P of a given configuration by the horizontal circle of center P whose diameter equals half the distance between P and the rest of the points and circles of the configuration.

Proposition 5.1 *The above map identifies $\mathbf{wP}(k,m)$ with the subgroup of $\mathbf{wP}_{k+m}$ generated by the $\chi_{\alpha\beta}$, for $1 \leqslant \alpha \neq \beta \leqslant n$ such that $\beta > k$.*

Proof. Using Lemma 5.2 below, we get a commutative diagram with (split) short exact rows:

$$
\begin{array}{ccccc}
\mathbf{P}_k(\mathbb{D}^3_m) & \lhook\joinrel\longrightarrow & \mathbf{wP}(k,m) & \longtwoheadrightarrow & \mathbf{wP}_m \\
\downarrow & & \downarrow & & \| \\
\bullet & \lhook\joinrel\longrightarrow & \mathbf{wP}_{k+m} & \longtwoheadrightarrow & \mathbf{wP}_m,
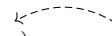\end{array}
$$

where the left-square is induced by the obvious commutative square at the level of configuration spaces. We want to show that the middle vertical map is injective. Using the Five Lemma, we see that we only need to show that the left vertical map is injective or, equivalently, that the dashed one is. Recall that $\mathbf{P}_k(\mathbb{D}^3_m) \cong (\mathbf{F}_m)^k$ via $\pi_{\mathbb{D}^3_m}$; see Proposition 6.9 below. Moreover, by unravelling the definitions, we see that our map $\mathbf{P}_k(\mathbb{D}^3_m) \to \mathbf{wP}_{k+m}$ sends the generator x_j of the i-th copy of $\mathbf{F}_n$ (corresponding to the i-th point passing through the j-th circle, with $i \leqslant k$ and $k < j \leqslant k+m$) to $\chi_{ij} \in \mathbf{wP}_{k+m}$. It follows directly from the interpretation of $\mathbf{wP}_{k+m}$ as a group of automorphism of the free group $\mathbf{F}_{k+m}$ (see Theorem A.12) that for fixed $i \leqslant k$, the χ_{ij} for all $k < j \leqslant k+m$ generate a free group, and that these copies of $\mathbf{F}_m$ commute in $\mathbf{wP}_{k+m}$, so that our map must be an isomorphism onto its image. As a consequence the map $\mathbf{wP}(k,m) \to \mathbf{wP}_{k+m}$ under scrutiny is injective.

Now, let G be the subgroup of $\mathbf{wP}_{k+m}$ generated by the $\chi_{\alpha\beta}$, for $1 \leqslant \alpha \neq \beta \leqslant n$ such that $\beta > k$. All these elements are seen to be images of elements of $\mathbf{wP}(k,m)$, so that $G \subseteq \mathbf{wP}(k,m)$. Moreover, G contains the image of $\mathbf{P}_k(\mathbb{D}^3_m)$ (described explicitly in the above reasoning), and all of $\mathbf{wP}_m$, so that G is equal to the whole of $\mathbf{wP}(k,m) = \mathbf{P}_k(\mathbb{D}^3_m) \rtimes \mathbf{wP}_m$, and the proposition is proved. $\qquad\square$

We denote by $\mathbb{D}^3_n$ the complement of an n-component unlink in the interior of the unit 3-disc.

Lemma 5.2 *The canonical map $F_{k \sqcup m \mathbb{S}^1}(\mathbb{D}^3) \to F_{m\mathbb{S}^1}(\mathbb{D}^3)$ induces a split short exact sequence:*

$$
\mathbf{P}_k(\mathbb{D}^3_m) \lhook\joinrel\longrightarrow \mathbf{wP}(k,m) \overset{\kappa}{\longtwoheadrightarrow} \mathbf{wP}_m,
$$

where the kernel $\mathbf{P}_k(\mathbb{D}^3_n)$ identifies with k copies of the free group $\pi_1(\mathbb{D}^3_m) \cong \mathbf{F}_m$.

Proof. By [Pal60, Th. C], the canonical map $p \colon F_{k \sqcup m \mathbb{S}^1}(\mathbb{D}^3) \to F_{m\mathbb{S}^1}(\mathbb{D}^3)$ is a locally trivial fibration. Its fibre identifies with $F_k(\mathbb{D}^3_m)$, where $\mathbb{D}^3_m$ denotes the (open) 3-ball with n unlinked, unknotted circles removed. Moreover, this fibration splits up to homotopy, that is, there exists

$s\colon F_{m\mathbb{S}^1}(\mathbb{D}^3) \to F_{k \sqcup m\mathbb{S}^1}(\mathbb{D}^3)$ such that $p \circ s \simeq id$. Namely, one can construct a (smooth) isotopy equivalence g between $\mathbb{D}^3$ and a proper subspace D of $\mathbb{D}^3$. This is an inverse up to isotopy of the inclusion $i_D\colon D \hookrightarrow \mathbb{D}^3$. Then one can fix a configuration c_0 of k points outside of D, and let s send a configuration c of circles to $g(c) \sqcup c_0$. The isotopy $i_D \circ g \simeq id$ then induces a homotopy $p \circ s \simeq id$, as required. As a consequence, the long exact sequence in homotopy breaks into split short exact sequences, and the one between fundamental groups is exactly the one of the lemma. Moreover, the identification of the kernel comes directly from the usual identification of braids on manifolds of dimension at least 3; see Proposition 6.9 below. $\qquad\square$

Corollary 5.3 *The abelianisation* $\mathbf{wP}(k,m)^{\mathrm{ab}}$ *is free abelian on the classes of the* $\chi_{\alpha\beta}$, *for* $1 \leqslant \alpha \neq \beta \leqslant n$ *such that* $\beta > k$.

Proof. By Proposition 5.1, these classes generate $\mathbf{wP}(k,m)^{\mathrm{ab}}$. Moreover, the canonical map $\mathbf{wP}(k,m)^{\mathrm{ab}} \to \mathbf{wP}_{k+m}^{\mathrm{ab}}$ sends these generators to linearly independent elements of $\mathbf{wP}_{k+m}^{\mathrm{ab}}$ (see Proposition A.8), so that they are a basis of $\mathbf{wP}(k,m)^{\mathrm{ab}}$. $\qquad\square$

5.2 Tripartite welded braids

Given three integers n_P, n_{S_+} and n_S, we define the group of *tripartite welded braids* by:

$$\mathbf{wB}(n_P, n_{S_+}, n_S) := \pi_1\left(F_{n_P \sqcup (n_{S_+} + n_S)\mathbb{S}^1}(\mathbb{D}^3)/(\mathfrak{S}_{n_P} \times \mathfrak{S}_{n_{S_+}} \times W_{n_S})\right).$$

In other words, we consider the fundamental group of the configuration space of n_P points, n_{S_+} oriented circles and n_S unoriented ones, where all the circles are supposed unlinked and unknotted.

In order to study it, we identify this group with a subgroup of $\widetilde{\mathbf{w}}\mathbf{B}_n$, where $n = n_P + n_{S_+} + n_S$. Since $\widetilde{\mathbf{w}}\mathbf{B}_n$ identifies with a subgroup of $\mathrm{Aut}(\mathbf{F}_n)$ (see Theorem A.12), we will then be able to identify tripartite welded braids with automorphisms of $\mathbf{F}_n$, and this point of view will come in handy for doing explicit computations.

Let us consider the map from $F_{n_P \sqcup (n_{S_+} + n_S)\mathbb{S}^1}(\mathbb{D}^3)$ to $F_{n\mathbb{S}^1}(\mathbb{D}^3)$ defined, as above, by replacing each point in a configuration by a small oriented circle. By factorisation through quotients by the appropriate group actions, we get a map between the configuration spaces of which $\mathbf{wB}(n_P, n_{S_+}, n_S)$ and $\widetilde{\mathbf{w}}\mathbf{B}_n$ are the fundamental groups (this boils down to replacing points in configurations by small circles, and forgetting orientations of circles).

Proposition 5.4 *The above map identifies* $\mathbf{wB}(n_P, n_{S_+}, n_S)$ *with the subgroup of* $\widetilde{\mathbf{w}}\mathbf{B}_n$ *with* $n = n_P + n_{S_+} + n_S$ *generated by:*

- *The* τ_α *for* $\alpha \neq n_P, n_P + n_{S_+}$ *and* $1 \leqslant \alpha < n$.
- *The* ρ_α *for* $n_P + n_{S_+} < \alpha \leqslant n$.
- *The* $\chi_{\alpha\beta}$, *for* $1 \leqslant \alpha \neq \beta \leqslant n$ *such that* $\beta > n_P$.

Proof. Consider the following commutative square of maps between configuration spaces:

$$
\begin{array}{ccc}
F_{n_P,(n_{S_+}+n_S)\mathbb{S}^1}(\mathbb{D}^3) & \longrightarrow & F_{n_P,(n_{S_+}+n_S)\mathbb{S}^1}(\mathbb{D}^3)/(\mathfrak{S}_{n_P} \times \mathfrak{S}_{n_{S_+}} \times W_{n_S}) \\
\downarrow & & \downarrow \\
F_{n\mathbb{S}^1}(\mathbb{D}^3) & \longrightarrow & F_{n\mathbb{S}^1}(\mathbb{D}^3)/W_n,
\end{array}
$$

where the horizontal maps are regular coverings. These induce the left square of the following commutative diagram, whose rows are exact (thanks to the usual theory of coverings):

$$
\begin{array}{ccccc}
\mathbf{wP}(n_P, n_{S_+} + n_S) & \longhookrightarrow & \mathbf{wB}(n_P, n_{S_+}, n_S) & \longrightarrow\!\!\!\!\!\rightarrow & \mathfrak{S}_{n_P} \times \mathfrak{S}_{n_{S_+}} \times W_{n_S} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{wP}_n & \longhookrightarrow & \widetilde{\mathbf{w}}\mathbf{B}_n & \longrightarrow\!\!\!\!\!\rightarrow & W_n,
\end{array}
$$

The vertical map on the right is obviously injective. By Lemma 5.1, the one on the left is also injective. So, by the Five Lemma, the middle vertical map $\mathbf{wB}(n_P, n_{S_+}, n_S) \to \widetilde{\mathbf{w}}\mathbf{B}_n$ is too.

Now, let G be the subgroup of $\widetilde{\mathbf{w}}\mathbf{B}_n$ generated by the elements listed in the statement. First, one easily sees that each element in the list is the image of some element of $\mathbf{wB}(n_P, n_{S_+}, n_S)$, so that $G \subseteq \mathbf{wB}(n_P, n_{S_+}, n_S)$. Moreover, G contains $\mathbf{wP}(n_P, n_{S_+} + n_S)$ by Proposition 5.1, and it also contains $\mathfrak{S}_{n_P} \times \mathfrak{S}_{n_{S_+}} \times W_{n_S}$, which is generated by the τ_α and the ρ_α contained in G. Thus, G is all of their semi-direct product $\mathbf{wB}(n_P, n_{S_+}, n_S)$, and our statement is proved. $\square$

5.3 Partitioned tripartite welded braids

Let us now introduce a partitioned version of this group. In order to do this, we use the canonical projection π from $\mathbf{wB}(n_P, n_{S_+}, n_S)$ onto $\mathfrak{S}_{n_P} \times \mathfrak{S}_{n_{S_+}} \times \mathfrak{S}_{n_S}$, which identifies (using Proposition 5.4) with a restriction of $\pi : \widetilde{\mathbf{w}}\mathbf{B}_n \twoheadrightarrow \mathfrak{S}_n$.

Definition 5.5 Let λ_P, λ_{S_+} and λ_S be partitions of integers n_P, n_{S_+} and n_S respectively. Let $\lambda = \lambda_P \lambda_{S_+} \lambda_S$ be their concatenation, which is a partition of $n = n_P + n_{S_+} + n_S$. The blocks of λ will often be identified with blocks of λ_P, λ_{S_+} and λ_S. The associated group of *tripartite welded braids* $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ is defined by:

$$\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S) = \pi^{-1}(\mathfrak{S}_\lambda) = \pi^{-1}(\mathfrak{S}_{\lambda_P} \times \mathfrak{S}_{\lambda_{S_+}} \times \mathfrak{S}_{\lambda_S}) \subseteq \mathbf{wB}(n_P, n_{S_+}, n_S) \subset \widetilde{\mathbf{w}}\mathbf{B}_n.$$

A direct adaptation of the proof of Lemma 3.3 to this context gives:

Lemma 5.6 *The group $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ identifies with the subgroup of $\widetilde{\mathbf{w}}\mathbf{B}_n$ generated by:*

- *The τ_α for $1 \leqslant \alpha < n$ such that α and $\alpha + 1$ are in the same block of λ.*
- *The σ_α for $n_P < \alpha < n$ such that α and $\alpha + 1$ are in the same block of λ.*
- *The ρ_α for $n_P + n_{S_+} < \alpha \leqslant n$.*
- *The $\chi_{\alpha\beta}$, for $1 \leqslant \alpha \neq \beta \leqslant n$ with $\beta > n_P$, such that α and β are not in the same block of λ.*

We can compute the abelianisation of the group $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$, like we did for partitioned groups before:

Proposition 5.7 *Let λ_P, λ_{S_+} and λ_S be partitions of integers n_P, n_{S_+} and n_S respectively, of respective length l_P, l_{S_+} and l_S. Let us denote by l'_P, l'_{S_+} and l'_S the number of blocks of size at least two in each of these partitions. Let also $\lambda_P \lambda_{S_+} \lambda_S = \lambda = (n_1, ..., n_l)$ be their concatenation, of length $l = l_P + l_{S_+} + l_S$, and with $l' = l'_P + l'_{S_+} + l'_S$ blocks of size at least two. Then:*

$$\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)^{\mathrm{ab}} \cong \mathbb{Z}^N \times (\mathbb{Z}/2)^M,$$

where $N = l'_{S_+} + l_{S_+}(l - 1)$ and $M = l' + l'_S + l_S l$.

Let us denote by I_P the set $\{1, ..., l_P\}$ of indices corresponding to blocks of λ_P, by I_{S_+} the set $\{l_P + 1, ..., l_P + l_{S_+}\}$ and by I_S the set $\{l_P + l_{S_+} + 1, ..., l\}$. With these notations, a basis of the first factor is given by:

- *For each $i \in I_{S+}$ such that $n_i \geqslant 2$, one generator s_i,*
- *For each $j \in I_{S+}$ and each $i \in \{1, ..., l\}$ such that $i \neq j$, one generator x_{ij}.*

and a $\mathbb{Z}/2$-basis of the second factor by:

- *For each $i \in I_S$ such that $n_i \geqslant 2$, one generator s_i,*
- *For each $i \in \{1, ..., l\}$ such that $n_i \geqslant 2$, one generator t_i,*
- *For each $i \in I_S$, one generator r_i,*
- *For each $j \in I_S$ and each $i \in \{1, ..., l\}$ such that $i \neq j$, one generator x_{ij}.*

The generators are obtained as follows:

- *s_i (resp. t_i) is the common class of the σ_α (resp. τ_α) for α and $\alpha + 1$ in the i-th block of λ.*

- r_i is the common class of the ρ_α for α in the i-th block of λ.
- x_{ij} is the common class of the $\chi_{\alpha\beta}$ for α in the i-th block of λ and β in the j-th one.

Proof. The proof is very similar to that of Proposition 4.21, which is a particular case of the present general statement. Namely, we apply Lemma 1.9 to the decomposition:

$$\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S) \cong \mathbf{wP}(n_P, n_{S_+} + n_S) \rtimes (\mathfrak{S}_{\lambda_P} \times \mathfrak{S}_{\lambda_{S_+}} \times W_{\lambda_S}).$$

The abelianisation of the second factor identifies with $(\mathbb{Z}/2)^{l'+l_S}$, generated by the generators t_i and r_i of the statement. The abelianisation of $\mathbf{wP}(n_P, n_{S_+} + n_S)$ is free on the classes of the $\chi_{\alpha\beta}$ with $1 \leqslant \alpha \neq \beta \leqslant n$ such that $\beta > n_P$ by Corollary 5.3, and the action of $\mathfrak{S}_{\lambda_P} \times \mathfrak{S}_{\lambda_{S_+}} \times W_{\lambda_S}$ is by permutation of the indices. We note that this latter group identifies with $(\mathbb{Z}/2)^{n_S} \rtimes \mathfrak{S}_\lambda$, where $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_P} \times \mathfrak{S}_{\lambda_{S_+}} \times \mathfrak{S}_{\lambda_S}$ acts on $(\mathbb{Z}/2)^{n_S}$ through the projection on its third factor (whose action permutes the factors). Thus, the coinvariants we need to compute are:

$$\left(\mathbf{wP}(n_P, n_{S_+} + n_S)^{\mathrm{ab}}\right)_{(\mathbb{Z}/2)^{n_S} \rtimes \mathfrak{S}_\lambda} \cong \left(\left(\mathbf{wP}(n_P, n_{S_+} + n_S)^{\mathrm{ab}}\right)_{(\mathbb{Z}/2)^{n_S}}\right)_{\mathfrak{S}_\lambda}.$$

The computation then continues exactly as in the proof of Proposition 4.21. In particular, $\left(\mathbf{wP}(n_P, n_{S_+} + n_S)^{\mathrm{ab}}\right)_{(\mathbb{Z}/2)^{n_S}}$ is a product of factors $\mathbb{Z}$ and $\mathbb{Z}/2$ generated by the $\overline{\chi}_{\alpha\beta}$, respectively with $n_P < \beta \leqslant n_P + n_{S_+}$ and with $\beta > n_P + n_{S_+}$. From the classes of these generators in the coinvariants, we get the generators x_{ij} of the statement, and the generators s_i, by the same change of basis as in the proof of Proposition 4.21. $\square$

From a geometrical point of view, the generators described in Proposition 5.7 correspond to the motions of points and circles in the 3-ball represented in Figure 5.1.
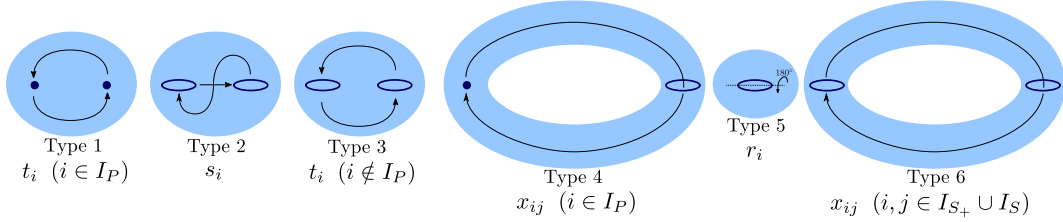

Figure 5.1   The six types of generators for $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)^{\mathrm{ab}}$ from Proposition 5.7. Types 2–4 each have two sub-types depending on whether the circles are (both) oriented or (both) unoriented. The circle in type 5 must be unoriented. Type 6 has four sub-types, depending on which of the circles are oriented. Type 2 generates a $\mathbb{Z}$ summand if both circles are oriented. Types 4 and 6 generate a $\mathbb{Z}$ summand if the non-moving circle is oriented. All other types generate a $\mathbb{Z}/2$ summand.

5.4   The lower central series

We now study the lower central series of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$. In order to do so, we need to apply disjoint support arguments. These arguments are best understood by thinking of motions having disjoint support in the 3-ball, and the reader is advised to keep this point of view in mind. However, writing down precise arguments and explicit calculations is much easier when dealing with automorphisms of free groups, so we mainly identify elements of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ with automorphisms of free groups in our proofs, using Proposition 5.4 and Theorem A.12.

The remainder of this section is devoted to the proof of the following theorem:

Theorem 5.8 *Let λ_P, λ_{S_+} and λ_S denote partitions of integers n_P, n_{S_+} and n_S respectively, of respective lengths l_P, l_{S_+} and l_S. Let also $\lambda_P \lambda_{S_+} \lambda_S = \lambda = (n_1, \ldots, n_l)$ be their concatenation, of length $l = l_P + l_{S_+} + l_S$. We denote by I_P the set $\{1, \ldots, l_P\}$ of indices corresponding to blocks of λ_P, by I_{S_+} the set $\{l_P + 1, \ldots, l_P + l_{S_+}\}$ and by I_S the set $\{l_P + l_{S_+} + 1, \ldots, l\}$.*

Let us suppose that $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ is not trivial, i.e. $(\lambda_P, \lambda_{S_+}, \lambda_S)$ is not among $(\varnothing, \varnothing, \varnothing)$, $((1, 1, ..., 1), \varnothing, \varnothing)$ and $(\varnothing, 1, \varnothing)$. The lower central series of the group $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$:

- stops at Γ_2 *if* $n_i \neq 2$ *for all* $i \in I_P$ *and* $n_i \geqslant 4$ *for all* $i \in I_{S_+} \cup I_S$, *save for at most one* $i \in I_{S_+}$ *such that* $n_i = 1$.
- stops at Γ_3 *if* $n_i = 2$ *for at least one* $i \in I_P$, $I_{S_+} = \varnothing$, $I_S \neq \varnothing$ *and* $n_i \geqslant 4$ *for all* $i \in I_S$.
- does not stop *in all the other cases, except for* $\mathbf{wB}(\varnothing, \varnothing, 1) = \widetilde{\mathbf{w}}\mathbf{B}_1 \cong \mathbb{Z}/2$.

Proof. The first statement is Proposition 5.12, which is a generalisation of Propositions 5.9, 5.10 and 5.11. The second one is Proposition 5.17.

Among the remaining cases, here are the ones for which we can deduce our conclusion directly from the calculations above (Propositions 4.11 and 4.16), by exhibiting a quotient of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ whose lower central series does not stop:

- If λ_S has one block of size 2 or 3, then it surjects onto $\widetilde{\mathbf{w}}\mathbf{B}_2$ or $\widetilde{\mathbf{w}}\mathbf{B}_3$.
- If λ_S has at least two blocks of size 1, then it surjects onto $\widetilde{\mathbf{w}}\mathbf{B}_{1,1} = \widetilde{\mathbf{w}}\mathbf{P}_2 \cong \mathbf{F}_2 \rtimes (\mathbb{Z}/2)^2$.
- If λ_{S_+} has one block of size 2 or 3, then it surjects onto $\mathbf{wB}_2$ or $\mathbf{wB}_3$.
- If λ_{S_+} has at least two blocks of size 1, then it surjects onto $\mathbf{wB}_{1,1} = \mathbf{wP}_2$.

In all of these cases, Lemma 1.1 allows us to conclude that the lower central series of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ does not stop.

Moreover, if λ_S has one block of size 1 which is not the only block of λ, then Proposition 5.13 ensures that the lower central series of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ does not stop. Then the only remaining case is the one where λ_P has at least one block of size 2, I_{S_+} is not empty and the blocks of λ_S are of size at least 4. This last case is covered by Proposition 5.14. $\qquad\square$

5.4.1 The stable case

We begin with the cases where all the blocks of λ are big enough to apply a disjoint support argument:

Proposition 5.9 *The notations being as in Theorem 5.8, suppose that $n_i \geqslant 3$ for all $i \in I_P$ and $n_i \geqslant 4$ for all $i \in I_{S_+} \cup I_S$. Then the lower central series of the group $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ stops at Γ_2.*

Proof. The proof is an extension of that of Theorem 4.22. By Corollary 2.6, we just have to show that any pair of the generators of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)^{\mathrm{ab}}$ listed in Proposition 5.7 (and illustrated in Figure 5.1) has a pair of lifts in $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ with disjoint support. The assumption that $n_i \geqslant 3$ for all $i \in I_P$ and $n_i \geqslant 4$ for all $i \in I_{S_+} \cup I_S$ is exactly what we need for that, as one sees by direct inspection. For instance, a generator x_{ij} has a lift whose support is $\{\alpha, \beta\}$ (namely, $\chi_{\alpha\beta}$), for any choice of α in the i-th block and β in the j-th block of λ. Thus any x_{kl} has a lift whose support is disjoint from some lift of x_{ij}, provided that the i-th and j-th blocks of λ both have size at least 2. $\qquad\square$

5.4.2 Blocks of points and oriented circles of size 1

The result in the stable case of Proposition 5.9 can be extended to the case where λ_P has blocks of size 1:

Proposition 5.10 *The notations being those of Theorem 5.8, if $n_i \geqslant 4$ for all $i \in I_{S_+} \cup I_S$ and λ_P has no blocks of size 2, then the lower central series of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ stops at Γ_2.*

Proof. We argue like in the proof of Proposition 4.24.

Let us write $\lambda_P = (1, \dots, 1, \mu_P)$, where there are k ones and each block of μ_P has size at least 3. There is an obvious homomorphism $\mathbf{wB}(\mu_P, \lambda_{S_+}, \lambda_S) \to \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$, which induces a homomorphism

$$\mathbf{wB}(\mu_P, \lambda_{S_+}, \lambda_S)^{\mathrm{ab}} = \mathbf{wB}(\mu_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty \longrightarrow \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty. \tag{5.1}$$

The equality on the left-hand side comes from Proposition 5.9. Similarly, for each block i of the partition $\lambda_{S_+} \cup \lambda_S$, there is an obvious homomorphism of the form $\mathbf{wB}((1,\ldots,1), n_i, \varnothing) \to \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ or $\mathbf{wB}((1,\ldots,1), \varnothing, n_i) \to \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$, inducing a homomorphism

$$
\begin{aligned}
&\mathbf{wB}((1,\ldots,1), n_i, \varnothing)^{\mathrm{ab}} = \mathbf{wB}((1,\ldots,1), n_i, \varnothing)/\Gamma_\infty \longrightarrow \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty \\
\text{or} \quad &\mathbf{wB}((1,\ldots,1), \varnothing, n_i)^{\mathrm{ab}} = \mathbf{wB}((1,\ldots,1), \varnothing, n_i)/\Gamma_\infty \longrightarrow \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty.
\end{aligned}
\tag{5.2}
$$

The equality on the left-hand side comes from the fact that the disjoint support argument used in the proof of Proposition 5.9 also works for showing that $\Gamma_2 = \Gamma_\infty$ in this particular case where there is only one block of circles.

Let us consider the generators $\tau_\alpha, \sigma_\alpha, \rho_\alpha, \chi_{\alpha\beta}$ of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ from Lemma 5.6. Using Proposition 5.7, we deduce from the existence of the morphisms (5.1) and (5.2) that their classes modulo Γ_∞ depend only on the blocks to which α and β belong. In other words: if α and α' belong to the same block of the partition, then $\sigma_\alpha = \sigma_{\alpha'}$ in $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty$, etc. Once we know this, it is easy to use a disjoint support argument to see that all of these generators commute pairwise, with the possible exception of $\chi_{\alpha\beta}$ and $\chi_{\alpha\beta'}$, where $1 \leqslant \alpha \leqslant k$ (so it corresponds to a block of points of size 1) and where β and β' lie in distinct blocks of $\lambda_{S_+} \cup \lambda_S$. We deal with these as follows. Let us suppose that β and β' both lie in λ_{S_+} (the argument in the other cases is similar). Consider the "adding strands" homomorphism

$$
\mathbf{wB}(1, (1,1), \varnothing) \longrightarrow \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S) \twoheadrightarrow \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty
\tag{5.3}
$$

induced by $1 \mapsto \alpha$, $2 \mapsto \beta$ and $3 \mapsto \beta'$. We know that the images of the elements χ_{13} and χ_{23} of the domain commute in $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty$. Thus the homomorphism (5.3) factors through the quotient of the left-hand side by $[\chi_{12}, \chi_{23}]$. Then Lemma 4.25 implies that the image $\chi_{\alpha\beta}$ of χ_{12} commutes with the image $\chi_{\alpha\beta'}$ of χ_{13} in $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty$. We have now shown that all the generators of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty$ commute pairwise, so it is abelian. Thus $\Gamma_2 = \Gamma_\infty$ for $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$. $\qquad\square$

The result in the stable case (Proposition 5.9) can also be extended to the case where λ_{S_+} has one block of size 1, thus generalising Proposition 4.24:

Proposition 5.11 *The notations being as in Theorem 5.8, suppose that $n_i \geqslant 3$ for all $i \in I_P$ and $n_i \geqslant 4$ for all $i \in I_{S_+} \cup I_S$, save for one $i \in I_{S_+}$ such that $n_i = 1$. Then the lower central series of the group $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ stops at Γ_2.*

Proof. Up to some permutation of the blocks, we can suppose that $\lambda_{S_+} = (1, \mu)$ for some partition μ whose blocks have size at least 4. Then, as in the proof of Proposition 4.24, we have a morphism:

$$
\mathbf{wB}(\lambda_P, \mu, \lambda_S)^{\mathrm{ab}} = \mathbf{wB}(\lambda_P, \mu, \lambda_S)/\Gamma_\infty \to \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty,
$$

where the first equality follows from Proposition 5.9. Starting from this, the generalisation of the proof of Proposition 4.24 to this context is straightforward. $\qquad\square$

In fact, the same result holds when there are both some blocks of points of size 1 and one block of oriented circles of size 1, the size of the other blocks being in the stable range:

Proposition 5.12 *The notations being as in Theorem 5.8, suppose that $n_i \neq 2$ for all $i \in I_P$, and $n_i \geqslant 4$ for all $i \in I_{S_+} \cup I_S$, save for one $i \in I_{S_+}$ such that $n_i = 1$. Then the lower central series of the group $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ stops at Γ_2.*

Proof. Let us denote by μ_P the partition obtained from λ_P by removing blocks of size 1, and by μ_{S_+} the one obtained from λ_{S_+} by removing the block of size 1. Then we have two morphisms:

$$
\begin{aligned}
\mathbf{wB}(\mu_P, \lambda_{S_+}, \lambda_S)^{\mathrm{ab}} = \mathbf{wB}(\mu_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty \longrightarrow \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty, \\
\mathbf{wB}(\lambda_P, \mu_{S_+}, \lambda_S)^{\mathrm{ab}} = \mathbf{wB}(\lambda_P, \mu_{S_+}, \lambda_S)/\Gamma_\infty \longrightarrow \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty,
\end{aligned}
$$

where the equalities on the left come from Propositions 5.10 and 5.11. As a consequence, the classes of the $\chi_{\alpha\beta}$, the τ_α, the σ_α and the ρ_α generating $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ in the quotient by Γ_∞ depend only on the blocks containing α and β. Indeed, this is obvious when α and β both belong to blocks of size one, which is the only case where it cannot directly be deduced from the above description of the abelianisations (Proposition 5.7). As usual, when α is in the i-th block and β in the j-th one, we denote these classes by x_{ij}, t_i, s_i and r_i respectively.

Now, let us consider two generators among these. Most of the time, both of them belong to the image of one of the morphisms above, whence they commute. When this does not hold, a disjoint support argument shows that they still commute, except when one of them is x_{ij} for $i \in I_P$ and $j \in I_{S_+}$ such that $n_i = n_j = 1$, and the other one is either x_{ik}, x_{jk} or x_{kj}, for some $k \in I_{S_+} \cup I_S$ different from j. We now show that these commute too, reasoning as in the proof of Proposition 4.24. Let us denote by α (resp. β) the only element of the i-th block (resp. the j-th one), and by γ, γ' two distinct elements of the k-th block (which must be of size at least 4).

- x_{ij} and x_{kj} always commute, since $\chi_{\alpha\beta}$ commutes with $\chi_{\gamma\beta}$.
- x_{ij} and x_{ik} are the images of χ_{13} and χ_{12} by a morphism from $\langle \chi_{12}, \chi_{23}, \chi_{13} \rangle$ sending χ_{23} to x_{kj}. Since x_{ik} and x_{kj} are the classes of $\chi_{\alpha\gamma}$ and $\chi_{\gamma'\beta}$, whose support is disjoint, they commute. Hence we can apply Lemma 4.25 to conclude that x_{ij} and x_{ik} commute.
- Finally, x_{ij} and x_{jk} are also in such a homomorphic image, and we know from the previous case that x_{ij} and x_{ik} commute, so we can apply Lemma 4.25 again to deduce that x_{ij} and x_{jk} commute too.

Finally, we have shown that all the generators of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)/\Gamma_\infty$ commute pairwise. Thus this group is abelian, which means that $\Gamma_\infty = \Gamma_2$ for $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$. $\qquad\square$

5.4.3 Small blocks of unoriented circles

We already know that if λ_S has at least one block of size 2 or 3, then $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ surjects onto $\widetilde{\mathbf{w}}\mathbf{B}_2$ or $\widetilde{\mathbf{w}}\mathbf{B}_3$, whose lower central series does not stop. The next proposition, which generalises Proposition 4.28, deals with the case where λ_S has at least one block of size 1:

Proposition 5.13 *Suppose that λ_S has at least one block of size 1, and there is at least another block in $\lambda = \lambda_P \lambda_{S_+} \lambda_S$. Then the lower central series of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ does not stop.*

Proof. The group $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ surjects either onto $\mathbf{wB}(m, 0, 1)$, onto $\mathbf{wB}(0, m, 1)$ or onto $\mathbf{wB}(0, 0, (m, 1))$, for some $m \geqslant 1$. We need to show that the lower central series of each of these three groups does not stop. The case of $\mathbf{wB}(0, 0, (m, 1)) = \widetilde{\mathbf{w}}\mathbf{B}_{1,m}$ has already been dealt with; see Proposition 4.28. The method used there can be adapted to the other two cases. Indeed, both $\mathbf{wB}(m, 0, 1)$ and $\mathbf{wB}(0, m, 1)$ identify with a subgroup of $\mathrm{Aut}(\mathbf{F}_{m+1})$ consisting of automorphisms preserving the normal closure N of the $x_\alpha x_{\alpha+1}^{-1}$ for $1 \leqslant \alpha \leqslant m - 1$, so they both project onto a subgroup G of $\mathrm{Aut}(\mathbf{F}_{m+1}/N) = \mathrm{Aut}(\mathbf{F}_2)$. In the first case, G is generated by ρ_2 and χ_{12}, and is isomorphic to $\mathbb{Z} \rtimes (\mathbb{Z}/2)$, whose lower central series does not stop by Proposition B.16. In the second case, G is generated by ρ_2, χ_{12} and χ_{21}. Then $\langle \chi_{12}, \chi_{21} \rangle = \mathrm{Inn}(\mathbf{F}_2) \cong \mathbf{F}_2$ is a normal subgroup of G, and:

$$G = \langle \chi_{12}, \chi_{21} \rangle \rtimes \langle \rho_2 \rangle \cong \mathbf{F}_2 \rtimes (\mathbb{Z}/2),$$

where the action of ρ_2 fixes χ_{21} and sends χ_{12} to its inverse. Then the quotient of G by χ_{21} is isomorphic to $\mathbb{Z} \rtimes (\mathbb{Z}/2)$, whose lower central series does not stop by Proposition B.16. $\qquad\square$

5.4.4 Blocks of points of size 2 with oriented circles

We now turn to the cases with blocks of points of size 2 which have not already been covered. The following result takes care of a large part of them:

Proposition 5.14 *Suppose that λ_P has at least one block of size 2, and that $\lambda_{S_+} \neq \varnothing$. Then the lower central series of $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ does not stop.*

Proof. Since λ_{S_+} is non-empty, it contains some block of size $m \geqslant 1$. Then $\mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ surjects onto $\mathbf{wB}(2, m, 0)$, so it it enough to show that the lower central series of $\mathbf{wB}(2, m, 0)$ does not stop. The latter is the subgroup of $\mathrm{Aut}(\mathbf{F}_{2+m})$ generated by τ_α and σ_α for $3 \leqslant \alpha \leqslant m+1$, $\chi_{1\alpha}$ and $\chi_{2\alpha}$ for $3 \leqslant \alpha \leqslant m+2$, and τ_1. All these automorphisms preserve the subgroup N normally generated by the $x_\alpha x_{\alpha+1}^{-1}$ for $3 \leqslant \alpha \leqslant m+1$, as one can check by direct inspection: for instance, $\sigma_\alpha(x_{\alpha-1}x_\alpha^{-1}) = x_{\alpha-1}(x_\alpha x_{\alpha+1} x_\alpha^{-1})^{-1} = (x_{\alpha-1}x_\alpha^{-1}) \cdot x_\alpha(x_\alpha x_{\alpha+1}^{-1})x_\alpha^{-1}$. As a consequence, they induce automorphisms of the quotient $\mathbf{F}_{2+m}/N \cong \mathbf{F}_3$. This defines a morphism from $\mathbf{wB}(2, m, 0)$ to $\mathrm{Aut}(\mathbf{F}_3)$ sending τ_α and σ_α to the identity, all the $\chi_{1\alpha}$ to χ_{13}, all the $\chi_{2\alpha}$ to χ_{23}, and τ_1 to τ_1. The image of this projection is thus $\langle \tau_1, \chi_{13}, \chi_{23} \rangle = \mathbf{wB}(2, 1, 0)$. Moreover, since $\chi_{23} = \tau_2\sigma_2$ and thus $\tau_1\chi_{23}\tau_1 = \chi_{13}$ (see §A.1.4), one easily sees that

$$\mathbf{wB}(2, 1, 0) = \langle \chi_{13}, \chi_{23} \rangle \rtimes \langle \tau_1 \rangle \cong \mathbb{Z}^2 \rtimes (\mathbb{Z}/2),$$

where the action of $\mathfrak{S}_2$, which is conjugation by τ_1, exchanges the two generators χ_{13} and χ_{23} of $\mathbb{Z}^2$. Thus, this semi-direct product identifies with $\mathbb{Z} \wr \mathfrak{S}_2$, whose lower central series does not stop (by Corollary B.27), so neither does the lower central series of $\mathbf{wB}(2, m, 0)$, as claimed. $\qquad\square$

Remark 5.15 One can give a more conceptual argument for the preservation of N by $\mathbf{wB}(2, m, 0)$. Namely, these automorphisms act on $\mathbf{F}_{2+m} \cong \mathbf{F}_2 * \mathbf{F}_m$ in a triangular fashion, and the automorphisms they induce on $\mathbf{F}_m$ preserve the characteristic subgroup $\Gamma_2(\mathbf{F}_m)$. Then their action on the quotient $\mathbf{F}_m^{\mathrm{ab}} \cong \mathbb{Z}^m$ factors through the canonical action of $\mathfrak{S}_m$, so the projection onto $\mathbb{Z}$ identifying all the generators of $\mathbf{F}_m$ is equivariant. Whence a well-defined induced action of $\mathbf{wB}(2, m, 0)$ on $\mathbf{F}_2 * \mathbb{Z} \cong \mathbf{F}_3$.

Remark 5.16 Concretely, the non-stopping of the lower central series in this case is witnessed by iterated commutators of elements of Type 4 and Type 1 from Figure 5.1. Here, we are viewing the pictures in Figure 5.1 as elements of the group $G = \mathbf{wB}(\lambda_P, \lambda_{S_+}, \lambda_S)$ itself, rather than as elements of its abelianisation $G^{\mathrm{ab}} = \mathcal{L}_1$.

5.4.5 Blocks of points of size 2 without oriented circles

We now consider the lower central series of $\mathbf{wB}(\lambda_P, \varnothing, \lambda_S)$. If there are no blocks in λ_S, then we are considering $\mathbf{wB}(\lambda_P, \varnothing, \varnothing) \cong \mathfrak{S}_{\lambda_P}$, whose lower central series stops at Γ_2 (except if $\lambda_P = (1, ..., 1)$, for which $\mathbf{wB}(\lambda_P, \varnothing, \varnothing) = \{1\}$). On the contrary, if there are no blocks in λ_P, then we are considering $\mathbf{wB}(\varnothing, \varnothing, \lambda_S) \cong \widetilde{\mathbf{w}}\mathbf{B}_{\lambda_P}$, whose lower central series has been studied above (in Proposition 4.22).

Suppose now that there is at least one block in both λ_P and λ_S. If λ_S has a block of size 1, 2 or 3, we already know that the lower central series of $\mathbf{wB}(\lambda_P, \varnothing, \lambda_S)$ does not stop (see §5.4.3); we can thus assume that the blocks of λ_S have size at least 4. Under this hypothesis, Proposition 5.10 takes care of the case where λ_P has no block of size 2 (then the lower central series stops). The remaining case is covered by the following.

Proposition 5.17 *Let λ_P be any partition having at least one block of size 2 and let λ_S be a non-empty partition whose blocks have size at least 4. Then the lower central series of $\mathbf{wB}(\lambda_P, \varnothing, \lambda_S)$ stops at Γ_3.*

Moreover, $\mathcal{L}_2(\mathbf{wB}(\lambda_P, \varnothing, \lambda_S))$ has a $(\mathbb{Z}/2)$-basis consisting of the $[t_i, x_{ij}]$, for $i \in I_P$ with $n_i = 2$, and $j \in I_S$, and all the other brackets of two of the generators of $\mathbf{wB}(\lambda_P, \varnothing, \lambda_S)^{\mathrm{ab}}$ described in Proposition 5.7 are trivial in the Lie ring.

Proof. Generators of $\mathcal{L}_1 = \mathbf{wB}(\lambda_P, \varnothing, \lambda_S)^{\mathrm{ab}}$ are described in Proposition 5.7 and Figure 5.1. Most of the pairs of such generators can be lifted to elements of $\mathbf{wB}(\lambda_P, \varnothing, \lambda_S)$ with disjoint support. The only ones for which this is not possible are:

- t_i and x_{ij} for $j \in I_S$ and $i \in I_P$ such that $n_i = 2$.
- x_{ij} and x_{ik} for $j, k \in I_S$ and $i \in I_P$ such that $n_i = 1$.

We first use Proposition 5.10 to show that $[x_{ij}, x_{ik}] = 0$ in the Lie ring. In order to do this, let us denote by μ_P the partition obtained from λ_P by removing the blocks of size 2. Proposition 5.10

implies that $\mathcal{L}(\mathbf{wB}(\mu_P, \varnothing, \lambda_S))$ is abelian. Since x_{ij} and x_{ik} are in the image of this Lie ring inside $\mathcal{L}(\mathbf{wB}(\lambda_P, \varnothing, \lambda_S))$, their bracket must be trivial.

We now know that $\mathcal{L}_2$ is generated by the commutator $z_{ij} := [t_i, x_{ij}]$, for $j \in I_S$ and $i \in I_P$ such that $n_i = 2$ (note that this also works when λ_P does not have blocks of size 2; then there are no z_{ij}, hence $\mathcal{L}_2 = 0$, and we recover the case $I_{S_+} = \varnothing$ of Proposition 5.10). Before going further in our analysis of the Lie ring, let us examine the element z_{ij} (with i, j as above). If the i-th block is $\{\alpha, \alpha + 1\}$ and β is in the j-th block, z_{ij} is the class of the commutator $\zeta_{\alpha\beta} := [\tau_\alpha, \chi_{\alpha\beta}]$, which is the automorphism:

$$\zeta_{\alpha\beta} : \begin{cases} x_\alpha & \longmapsto & x_\beta^{-1} x_\alpha x_\beta & \longmapsto & x_\beta^{-1} x_{\alpha+1} x_\beta & \longmapsto & x_\beta^{-1} x_{\alpha+1} x_\beta & \longmapsto & x_\beta^{-1} x_\alpha x_\beta, \\ x_{\alpha+1} & \longmapsto & x_{\alpha+1} & \longmapsto & x_\alpha & \longmapsto & x_\beta x_\alpha x_\beta^{-1} & \longmapsto & x_\beta x_{\alpha+1} x_\beta^{-1} \end{cases}$$

where all the x_γ with $\gamma \notin \{\alpha, \alpha + 1\}$ are fixed. The corresponding motion is drawn in Figure 5.2. We note that $\zeta_{\alpha\beta} = \chi_{\alpha+1,\beta} \chi_{\alpha\beta}^{-1}$ (which can be seen directly, or deduced from $\tau_\alpha \chi_{\alpha\beta} \tau_\alpha^{-1} = \chi_{\alpha+1,\beta}$).

Let us turn our attention to $\mathcal{L}_3$. It is generated by the commutators of the generators $z_{ij} = \overline{\zeta_{\alpha\beta}}$ of $\mathcal{L}_2$ with the generators of $\mathcal{L}_1$. Since $\mathrm{Supp}(\zeta_{\alpha\beta}) = \{\alpha, \alpha + 1, \beta\}$ (where β can be chosen anywhere in the j-th block for $\zeta_{\alpha\beta}$ to represent z_{ij}), one can choose a lift having disjoint support with $\zeta_{\alpha\beta}$ for each generator of $\mathcal{L}_1$, save x_{ik} (for $k \in I_S$) and t_i. If $k = j$, we remark that x_{ij} does have a lift commuting with $\zeta_{\alpha\beta}$, albeit not for reasons of support. Indeed, it is the class of $\chi_{\alpha\beta}$, which clearly commutes with $\zeta_{\alpha\beta}$, since both are automorphisms conjugating x_α and $x_{\alpha+1}$ by powers of the same element x_β. Thus the only generators of $\mathcal{L}_3$ which are possibly non-trivial are the $[t_i, z_{ij}]$ and the $[x_{ik}, z_{ij}]$ for $k \neq j$.

We first show that $[x_{ik}, z_{ij}]$ vanishes, when $k \neq j$. Let γ be in the k-th block of λ. Then $[x_{ik}, z_{ij}]$ is the class of the commutator $[\chi_{\alpha\gamma}, \zeta_{\alpha\beta}]$. Since $\zeta_{\alpha\beta} = \chi_{\alpha+1,\beta} \chi_{\alpha\beta}^{-1}$ and $\chi_{\alpha\gamma}$ commutes with $\chi_{\alpha+1,\beta}$, this commutator is equal to $[\chi_{\alpha\gamma}, \chi_{\alpha\beta}^{-1}]$. We have seen above that $\chi_{\alpha\gamma}$ commutes with $\chi_{\alpha\beta}$ modulo Γ_∞. Thus, the class of $[\chi_{\alpha\gamma}, \chi_{\alpha\beta}^{-1}]$ is trivial in the Lie ring, which means that $[x_{ik}, z_{ij}] = 0$.

We now show that $[t_i, z_{ij}] = 0$, using explicit computations. We have:

$$\tau_\alpha \zeta_{\alpha\beta} \tau_\alpha^{-1} : \begin{cases} x_\alpha & \longmapsto & x_{\alpha+1} & \longmapsto & x_\beta x_{\alpha+1} x_\beta^{-1} & \longmapsto & x_\beta x_\alpha x_\beta^{-1}, \\ x_{\alpha+1} & \longmapsto & x_\alpha & \longmapsto & x_\beta^{-1} x_\alpha x_\beta & \longmapsto & x_\beta^{-1} x_{\alpha+1} x_\beta \end{cases}$$

which means that $\tau_\alpha \zeta_{\alpha\beta} \tau_\alpha^{-1} = \zeta_{\alpha\beta}^{-1}$. We also have:

$$\rho_\beta \zeta_{\alpha\beta} \rho_\beta^{-1} : \begin{cases} x_\alpha & \longmapsto & x_\alpha & \longmapsto & x_\beta^{-1} x_\alpha x_\beta & \longmapsto & x_\beta x_\alpha x_\beta^{-1}, \\ x_{\alpha+1} & \longmapsto & x_{\alpha+1} & \longmapsto & x_\beta x_{\alpha+1} x_\beta^{-1} & \longmapsto & x_\beta^{-1} x_{\alpha+1} x_\beta \\ x_\beta & \longmapsto & x_\beta^{-1} & \longmapsto & x_\beta^{-1} & \longmapsto & x_\beta \end{cases}$$

whence $\rho_\beta \zeta_{\alpha\beta} \rho_\beta^{-1} = \zeta_{\alpha\beta}^{-1} = \tau_\alpha \zeta_{\alpha\beta} \tau_\alpha^{-1}$. Finally:

$$[t_i, z_{ij}] = \overline{[\tau_\alpha, \zeta_{\alpha\beta}]} = \overline{[\rho_\beta, \zeta_{\alpha\beta}]} = [r_j, z_{ij}] = 0,$$

where the last equality comes from the above disjoint support argument. This finishes the proof that $\mathcal{L}_3 = 0$.

We remark that the above calculation also implies $\Gamma_3 \ni [\tau_\alpha, \zeta_{\alpha\beta}] = \zeta_{\alpha\beta}^{-2}$, so that $z_{ij} = \overline{\zeta_{\alpha\beta}}$ is of order 2 in $\mathcal{L}_2$. Thus, in order to prove our last claim, we need to show that the z_{ij} are linearly independant over $\mathbb{Z}/2$. For each fixed choice of $j \in I_S$ and $i \in I_P$ such that $n_i = 2$, we have a projection from $\mathbf{wB}(\lambda_P, \varnothing, \lambda_S)$ onto $\mathbf{wB}(2, \varnothing, n_j)$ which induces a morphism between the $\mathcal{L}_2$ killing all the z_{kl} save z_{ij}, which is sent to z_{12}. Thus, we only need to show that for $n \geqslant 4$, the element z_{12} is non-trivial in $\mathcal{L}_2(\mathbf{wB}(2, \varnothing, n))$, which is then isomorphic to $\mathbb{Z}/2$. We do this by constructing one further projection. Elements of $\mathbf{wB}(2, \varnothing, n)$ are automorphisms of $\mathbf{F}_{2+n}$ preserving the normal subgroup generated by the $x_\alpha x_{\alpha+1}$ together with the x_α^2 ($\alpha \geqslant 3$). Whence a well-defined morphism $\mathbf{wB}(2, \varnothing, n) \to \mathrm{Aut}(\mathbf{F}_2 * (\mathbb{Z}/2))$. The image of this morphism is easily seen to be isomorphic to $W_2 = (\mathbb{Z}/2) \wr \mathfrak{S}_2$ (where the images of χ_{13} and χ_{23} generate $(\mathbb{Z}/2)^2$ and the image of τ_1 generates $\mathfrak{S}_2$), and the element $\zeta_{13} = \chi_{23} \chi_{13}^{-1}$ is sent to the generator $(1, 1, \mathrm{id})$ of $\Gamma_2(W_2)$ (see Proposition B.10 for the calculation of $\Gamma_*(W_2)$). Finally, the morphism induced between the Lie rings sends z_{12} to the generator of $\mathcal{L}_2(W_2) \cong \mathbb{Z}/2$, whence the result. $\qquad\square$
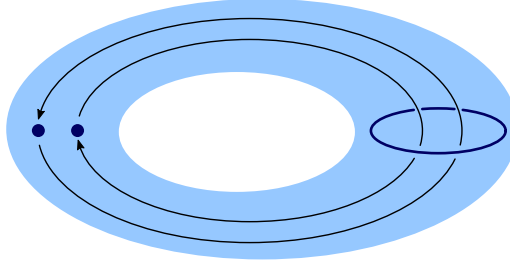
Figure 5.2   Generators of $\mathcal{L}_2 = \Gamma_2/\Gamma_3$ for $G = \mathbf{wB}(\lambda_P, \varnothing, m)$, where each block of λ_P has size $\geqslant 2$ and $m \geqslant 4$. There is one generator corresponding to the commutator $\zeta_{\alpha,\beta}$, represented by a loop of the form pictured, for each block of λ_P of size exactly 2. This generalises to $G = \mathbf{wB}(\lambda_P, \varnothing, \lambda_S)$ where each block of λ_S has size $\geqslant 4$, and the generators are indexed by (b, x) where b is a block of points of size exactly 2 and x is a block of circles.

6   Braids on surfaces

Here, we study the lower central series of surface braid groups and their partitioned versions. This may be seen as a generalisation of results from §3, where we studied classical Artin braids, that is, braids on the disc. This is even a wilder generalisation than the one generally found in the literature: we do not assume that our surfaces are compact or orientable. In particular, they may have (countably) infinite genus or boundary components. Among all these surface braid groups, four cases will stand out from the crowd: the braid groups on the sphere $\mathbb{S}^2$, on the torus $\mathbb{T}^2$, on the Möbius strip $\mathbb{M}^2$ and on the projective plane $\mathbb{P}^2$.

6.1   Surfaces

We recall that a *surface* is a (separable) 2-manifold, which we will *not* suppose compact or orientable in general. Such manifolds are well-understood: the classification of surfaces without boundary has been achieved by Richards [Ric63], and the classification of surfaces with boundary, which is more complicated, was completed more than fifteen years later by Brown and Messer [BM79].

Remark 6.1 All our manifolds are assumed separable. (And for the avoidance of ambiguity, we stipulate that *manifold* means a *locally Euclidean, Hausdorff space*.) We recall that separability is equivalent to triangulability; see [Rad25]. Moreover, separability is implied by second countability, which is equivalent, for connected manifolds, to either metrisability or paracompactness.

For studying braids, we will in fact only need to consider Richards' classification. Indeed, let S be any connected surface (possibly with boundary). If we glue a copy of $\partial S \times [0, \infty)$ to S by identifying $\partial S \times 0$ with ∂S, we obtain a surface S' without boundary, and it is easy to show that the canonical injection $S \hookrightarrow S'$ is an isotopy equivalence. Moreover, one can show that an isotopy equivalence induces homotopy equivalences between configuration spaces and, in turn, isomorphisms between braid groups. In the sequel, we thus identify braids on S with braids on S' (or on $S - \partial S$). For instance, braids on the closed disc are identified with braids on the plane. This holds in particular for braids on one strand, that is, for fundamental groups, whose computation is recalled below in Proposition 6.4.

Let us briefly recall Richard's construction of all surfaces, up to homeomorphism; for a detailed account, the reader is referred to [Ric63], in particular to §5 and §6, especially Theorem 3, therein.

Proposition 6.2 *Let S be a connected surface. Then S is homeomorphic to a surface constructed in the following way:*

- *Consider the Cantor set K embedded in the sphere $\mathbb{S}^2$ in the usual manner. Choose some closed subset X of K, and remove it from $\mathbb{S}^2$.*
- *Choose a finite or countably infinite sequence of pairwise disjoint closed 2-discs in $\mathbb{S}^2 - X$, which has no accumulation point outside of X.*
- *Along each of these discs, perform a connected sum operation with either $\mathbb{T}^2$ or $\mathbb{P}^2$.*

Remark 6.3 At the second step, one can in fact choose an explicit sequence of discs depending only on X together with the subset $X_{np} \subseteq X$ of accumulation points of the sequence of discs; see [Ric63].

As a direct corollary of Richards' classification, one can compute fundamental groups of surfaces:

Proposition 6.4 *Let S be a connected surface without boundary. Then $\pi_1(S)$ is a free group, except when S is closed. Moreover, it is of finite type if and only if S is obtained (up to homeomorphism) from a closed surface by removing a finite number of points.*

Remark 6.5 We see that closed surfaces (that is, compact surfaces without boundary) are singled out here, and it will be the case later in our study of braid groups; see for instance Proposition 6.25 and Theorem 6.30.

Another immediate corollary of Richards' classification is the following dichotomy:

Proposition 6.6 *Let S be a connected, separable surface without boundary. Then either S can be embedded into the sphere $\mathbb{S}^2$, or it contains the 1-punctured torus or the Möbius strip as an embedded subsurface.*

In our study of braids, this appears as a trichotomy, between the following cases:

- S is *planar*, i.e. it embeds into the plane;
- S is the sphere $\mathbb{S}^2$;
- S contains an embedded 1-punctured torus (a *handle*) or an embedded Möbius strip (a *crosscap*).

In that regard, see in particular Proposition 6.10.

6.2   Braids on surfaces: general theory

Let S be a connected surface. Let us consider the configuration space

$$F_n(S) = \{(x_1, ..., x_n) \in S^n \mid \forall i \neq j, \ x_i \neq x_j\} \subset S^n.$$

The *braid group on the surface S* on n strings is the fundamental group $\mathbf{B}_n(S)$ of the unordered configuration space $C_n(S) = F_n(S)/\mathfrak{S}_n$. When S is the 2-disc $\mathbb{D}$, this group is exactly Artin's braid group, that is $\mathbf{B}_n = \mathbf{B}_n(\mathbb{D})$.

Let us fix an embedded disc $\mathbb{D} \subset S$, together with a base configuration $c = (c_1, ..., c_n) \in F_n(S)$ of points $c_i \in \mathbb{D}$. Since the assignment $S \mapsto \mathbf{B}_n(S)$ is functorial with respect to embeddings of surfaces, we have a (not necessarily injective) group morphism:

$$\varphi : \mathbf{B}_n = \mathbf{B}_n(\mathbb{D}) \to \mathbf{B}_n(S).$$

In the sequel, we omit most mentions of φ: if $\beta \in \mathbf{B}_n$, we still denote by β its image in $\mathbf{B}_n(S)$, which we should denote by $\varphi(\beta)$.

We can also construct surface braids from curves on the surface. Precisely, for any $i \leqslant n$, let us define $\iota_i : S - \mathbb{D}^{\mathrm{o}} \hookrightarrow F_n(S)$ by sending x to $(c_1, ..., c_{i-1}, x, c_{i+1}, ..., c_n)$. This induces a morphism between fundamental groups:

$$\psi_i : \pi_1(S - \mathbb{D}^{\mathrm{o}}) \to \mathbf{P}_n(S).$$

Remark 6.7 The map ι_i cannot preserve basepoints, since the basepoint of $F_n(S)$ is not in its image. However, the induced map between fundamental groups can easily be defined using a chosen fixed path between c_i and a point not in $\mathbb{D}^{\mathrm{o}}$. The choice of such a path (for each i) is implicit in the sequel, and should be made as simply as possible. For instance, one can fix a segment from c_i to a point on $\partial\mathbb{D}$, such that these segments are disjoint for different values of i.

The canonical projection $\pi : \mathbf{B}_n(S) \twoheadrightarrow \mathfrak{S}_n$, corresponding to the covering of $F_n(S)/\mathfrak{S}_n$ by $F_n(S)$, can be enhanced to a projection:

$$\pi_S : \mathbf{B}_n(S) \twoheadrightarrow \pi_1(S) \wr \mathfrak{S}_n$$

as follows. Given a braid $\beta \in \mathbf{B}_n(S)$, let us lift it to a path $\gamma = (\gamma_1, ..., \gamma_n)$ in $F_n(S)$ from $(c_i)_i$ to $(c_{\sigma^{-1}(i)})_i$ where $\sigma = \pi(\beta)$. Then send β to $((\overline{\gamma}_1, ..., \overline{\gamma}_n), \sigma)$, where $\overline{\gamma}_i$ is the image of γ_i in $\pi_1(S/\mathbb{D}) \cong \pi_1(S)$. We note that π_S is clearly surjective, since its image contains $\mathfrak{S}_n$ (which is the image of $\varphi(\mathbf{B}_n)$ by π_S), and all the factors $\pi_1(S)$ (which are the images of the $\psi_i(\pi_1(S))$).

The kernel of π_S, which in contained in $\mathbf{P}_n(S)$, obviously contains the group $\mathbf{P}_n$. We denote it by $\mathbf{P}_n^\circ(S)$ and we call its elements *geometrically pure braids*. The following result generalises one of Goldberg [Gol73, Th. 1] to any connected surface.

Proposition 6.8 *The following statements hold for any connected surface S and any integer $n \geqslant 1$:*

- *For any $i \leqslant n$, the group $\mathbf{B}_n(S)$ is generated by the images of φ and ψ_i.*
- *Its subgroup $\mathbf{P}_n(S)$ is generated by (the image of) $\mathbf{P}_n$ and the images of $\psi_1, ..., \psi_n$.*
- *The subgroup $\mathbf{P}_n^\circ(S)$ of $\mathbf{P}_n(S)$ is the normal closure of $\mathbf{P}_n$. Since $\mathbf{P}_n^\circ(S)$ is normal in $\mathbf{B}_n(S)$, it is also the normal closure of $\mathbf{P}_n$ in $\mathbf{B}_n(S)$.*

Proof. Let us first remark that the ψ_i are conjugate to each other by elements of the image of φ. Hence, the second statement implies the first one.

We prove both the second and the third statement by induction on n. Both proofs use the tools that we introduce now. Consider the following commutative diagram of spaces:

$$
\begin{array}{ccccc}
S - Q_n & \overset{\iota}{\hookrightarrow} & F_{n+1}(S) & \overset{p}{\twoheadrightarrow} & F_n(S) \\
\downarrow & & \downarrow & & \downarrow \\
S & \overset{\iota}{\hookrightarrow} & S^{n+1} & \overset{p}{\twoheadrightarrow} & S^n,
\end{array}
$$

where $Q_n = \{c_1, ..., c_n\}$, ι sends x to $(c_1, ..., c_n, x)$ and p send $(x_1, ..., x_{n+1})$ to $(x_1, ..., x_n)$. It induces a commutative diagram of morphisms between fundamental groups:

$$
\begin{array}{ccccc}
\pi_1(S - Q_n) & \overset{\iota_*}{\longrightarrow} & \mathbf{P}_{n+1}(S) & \overset{p_*}{\twoheadrightarrow} & \mathbf{P}_n(S) \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\
\pi_1(S) & \hookrightarrow & \pi_1(S)^{n+1} & \longrightarrow & \pi_1(S)^n,
\end{array}
\tag{6.1}
$$

whose bottom line is obviously exact. The map p is a (locally trivial) fibration by [FN62, Th. 3] (see also [Bir74, Th. 1.2]), and the first line is part of its exact sequence in homotopy. Thus it is exact (but ι_* need not be injective in general).

Let us prove our second statement. For $n = 1$, ψ_1 is the canonical isomorphism $\pi_1(S) \cong \mathbf{P}_1(S) = \mathbf{B}_1(S)$, and there is nothing to prove.

Let us now suppose that the conclusion holds for some $n \geqslant 1$. By applying Van Kampen's theorem, we see that $\pi_1(S - Q_n) \cong \mathbf{F}_n *_\mathbb{Z} \pi_1(S - \mathbb{D}^\mathrm{o})$, where $\mathbf{F}_n = \pi_1(\mathbb{D} - Q_n)$ is free on n generators. Then $\iota_* : \mathbf{F}_n *_\mathbb{Z} \pi_1(S - \mathbb{D}^\mathrm{o}) \to \mathbf{P}_n(S)$ identifies with the map induced by $\mathbf{F}_n \hookrightarrow \mathbf{P}_n$ (kernel of $\mathbf{P}_n \twoheadrightarrow \mathbf{P}_{n-1}$) and $\psi_{n+1} : \pi_1(S - \mathbb{D}^\mathrm{o}) \to \mathbf{P}_n(S)$.

Now let G be the subgroup of $\mathbf{P}_{n+1}(S)$ generated by $\mathbf{P}_{n+1}$ and the images of the ψ_i for $i \leqslant n + 1$. It contains the image of ι_*, which is the kernel of p_*. Moreover, its image by p_* contains the images of $\psi_1, ..., \psi_n$, and $\mathbf{P}_n$, hence all of $\mathbf{P}_n(S)$, by the induction hypothesis. As a consequence, $G = \mathbf{P}_{n+1}(S)$, which was the desired conclusion.

Let us prove our third statement. For $n = 1$, π_S is the canonical isomorphism $\mathbf{B}_1(S) = \mathbf{P}_1(S) \cong \pi_1(S)$ (inverse to ψ_1). Then $\mathbf{P}_n^\circ(S)$ and $\mathbf{P}_n$ are trivial, and there is nothing to prove.

Let us now suppose that the conclusion holds for some $n \geqslant 1$. Consider the induced maps between kernels of the vertical morphisms in (6.1). By definition, the kernels of v and w are respectively $\mathbf{P}_{n+1}^\circ(S)$ and $\mathbf{P}_n^\circ(S)$. Let us denote by K the kernel of u. We get induced maps:

$$
K \overset{\iota_\#}{\longrightarrow} \mathbf{P}_{n+1}^\circ(S) \overset{p_\#}{\twoheadrightarrow} \mathbf{P}_n^\circ(S)
$$

such that $p_\# \circ \iota_\# = 1$. An easy chase in the diagram (or an application of the Snake Lemma) shows that we can lift any element in the kernel of $p_\#$ to an element of K: the above sequence is exact.

The morphism u identifies with the projection $\mathbf{F}_n *_{\mathbb{Z}} \pi_1(S - \mathbb{D}^\mathrm{o}) \twoheadrightarrow \{1\} *_{\mathbb{Z}} \pi_1(S - \mathbb{D}^\mathrm{o}) \cong \pi_1(S)$ killing the first factor, hence K identifies with the normal closure of $\mathbf{F}_n$ in $\mathbf{F}_n *_{\mathbb{Z}} \pi_1(S)$. Moreover, ι_* sends $\mathbf{F}_n = \pi_1(\mathbb{D} - Q_n)$ to a subgroup of $\mathbf{P}_{n+1}$, so the image of K in $\mathbf{P}_{n+1}(S)$ is contained in the normal closure of $\mathbf{P}_{n+1}$.

Now, let N be the normal closure of $\mathbf{P}_{n+1}$ in $\mathbf{P}_{n+1}(S)$. Since $v(\mathbf{P}_{n+1}) = \{1\}$, we have $v(N) = \{1\}$, which means that $N \subseteq \mathbf{P}_{n+1}^\circ(S)$. By the induction hypothesis, $p_\#(N) = \mathbf{P}_n^\circ(S)$. Moreover, N contains the image of $i_\#$, which is the kernel of $p_\#$. Thus $N = \mathbf{P}_{n+1}^\circ(S)$, which was the desired conclusion. $\qquad\square$

The proof of Proposition 6.8 also works for manifolds in higher dimension, allowing us to recover the classical [Bir69, Th. 1]:

Proposition 6.9 *For any manifold M of dimension at least 3, $\pi_M \colon \mathbf{B}_n(M) \twoheadrightarrow \pi_1(M) \wr \mathfrak{S}_n$ is an isomorphism.*

Proof. One can directly check that the proof of Proposition 6.8 works if we replace the surface S with a connected manifold M of any dimension $d \geqslant 2$ and the disc $\mathbb{D}$ with a d-disc $\mathbb{D}^d$. Then $\mathbf{P}_n$ gets replaced with $\mathbf{P}_n(\mathbb{D}^d)$, which is trivial whenever $d \geqslant 3$ (the configuration space $F_n(\mathbb{D}^d) \cong F_n(\mathbb{R}^d)$ is obtained from $\mathbb{R}^{nd}$ by removing subspaces of codimension $d \geqslant 3$, so it is simply connected). Thus, the normal closure $\mathbf{P}_n^\circ(M)$ of $\mathbf{P}_n(\mathbb{D}^d)$ is trivial too, and the latter is exactly the kernel of π_M. $\qquad\square$

This means that **braid groups on manifolds of dimension at least 3 are exactly wreath products**, whose lower central series in studied in the appendix; see in particular Corollaries B.25 and B.27.

Pure braid generators and commutators. Some of the results below will hold for all surfaces S. However, in order to get more precise results, we need to get more specific and use the classification of surfaces recalled in §6.1. Recall that all the generators σ_i of $\mathbf{B}_n$ are identified in $\mathbf{B}_n^{\mathrm{ab}} \cong \mathbb{Z}$ (see Example 2.3), hence also in $\mathbf{B}_n(S)^{\mathrm{ab}}$. The following Lemma deals notably with the order of their common class σ. The trichotomy that appears here, which comes from Proposition 6.6, will play an important role in all that follows:

Proposition 6.10 *Let $n \geqslant 2$. Let us consider the generator A_{ij} of $\mathbf{P}_n$ as an element of $\mathbf{B}_n(S)$.*

- *If S is planar, then the class $\overline{A}_{ij} \in \mathbf{B}_n(S)^{\mathrm{ab}}$ has infinite order.*
- *If $S \cong \mathbb{S}^2$, then the class $\overline{A}_{ij} \in \mathbf{B}_n(S)^{\mathrm{ab}}$ has order $n - 1$. However, its class in $\mathbf{P}_n(S)^{\mathrm{ab}}$ has infinite order.*
- *In all the other cases, A_{ij} is the commutator of two elements of $\mathbf{P}_n(S)$.*

Proof. **If S is planar:** then S can be embedded in a disc. This embedding induces a morphism $\mathbf{B}_n(S) \to \mathbf{B}_n(\mathbb{D}) = \mathbf{B}_n$, which in turn induces a morphism from $\mathbf{B}_n(S)^{\mathrm{ab}}$ to $\mathbf{B}_n^{\mathrm{ab}}$. The latter is infinite cyclic, generated by σ. Our element $\overline{A}_{ij}$ is sent to σ^2, hence it cannot be of finite order.

If S is the sphere: then from the usual presentation of $\mathbf{B}_n(\mathbb{S}^2)$ (see Corollary A.18), we get that $\mathbf{B}_n(\mathbb{S}^2)^{\mathrm{ab}} \cong \mathbb{Z}/(2(n - 1))$, generated by σ (which is the common class of the usual σ_i). Again, $\overline{A}_{ij} = \sigma^2$, whose order is $n - 1$.

If S cannot be embedded in the sphere: then S contains a handle or a crosscap; see Proposition 6.6 and the remark following it. We can then use the explicit isotopies drawn in Figure 6.1 and 6.2 to show that A_{ij} is a bracket of two pure braids (which are respectively in the image of ψ_j and in the image of ψ_i). $\qquad\square$
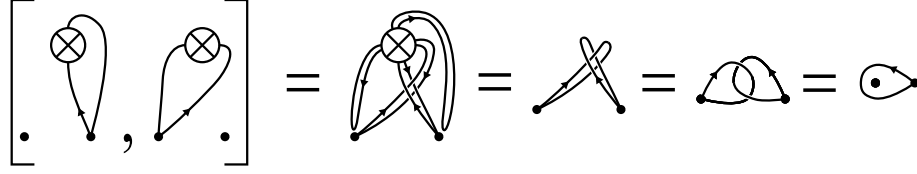
Figure 6.1   Pure braid generator as a bracket on a surface with a crosscap.
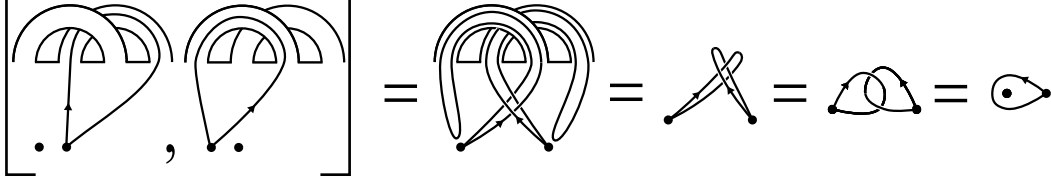

Figure 6.2   Pure braid generator as a bracket on a surface with a handle.

6.3   The lower central series of the whole group

We now turn to the study of the lower central series of $\mathbf{B}_n(S)$, which we completely determine for any $n \geqslant 3$ and any surface S. We begin by computing the abelianisation of this group; see Proposition 6.11. Then we study $\mathbf{B}_n(S)/\Gamma_\infty$, and we show that when $n \geqslant 3$, it is nilpotent of class at most 2, which means that the lower central series of $\mathbf{B}_n(S)$ stops at most at Γ_3; see Theorem 6.13 and Corollary 6.15. Finally, we compute the Lie ring, generalising a result of [BGG08]; see Theorem 6.19.

6.3.1   The abelianisation

We first compute the abelianisation of $\mathbf{B}_n(S)$, for any n and any surface S. In order to do this, we recall that the group morphism $\varphi \colon \mathbf{B}_n \to \mathbf{B}_n(S)$ induces a map $\mathbf{B}_n^{\mathrm{ab}} \to \mathbf{B}_n(S)^{\mathrm{ab}}$, and that, since all the σ_i are identified in $\mathbf{B}_n^{\mathrm{ab}} \cong \mathbb{Z}$, they are so also in $\mathbf{B}_n(S)^{\mathrm{ab}}$. We denote by σ their common image in $\mathbf{B}_n(S)^{\mathrm{ab}}$.

Proposition 6.11 *In general, for all $n \geqslant 2$, we have:*

$$\mathbf{B}_n(S)^{\mathrm{ab}} \cong \pi_1(S)^{\mathrm{ab}} \times \langle \sigma \rangle.$$

Moreover, σ is:

- *of infinite order if S is planar,*
- *of order $2(n-1)$ if $S \cong \mathbb{S}^2$,*
- *of order 2 in all the other cases.*

Proof. Applying Lemma 1.8 to the short exact sequence $\mathbf{P}_n^\circ(S) \hookrightarrow \mathbf{B}_n(S) \xrightarrow{\pi_S} \pi_1(S) \wr \mathfrak{S}_n$, we get an exact sequence of abelian groups:

$$(\mathbf{P}_n^\circ(S)^{\mathrm{ab}})_{\mathbf{B}_n(S)} \longrightarrow \mathbf{B}_n(S)^{\mathrm{ab}} \longrightarrow\!\!\!\!\rightarrow (\pi_1(S) \wr \mathfrak{S}_n)^{\mathrm{ab}} \longrightarrow 0.$$

On the one hand, the quotient $(\pi_1(S) \wr \mathfrak{S}_n)^{\mathrm{ab}}$ is isomorphic to $\pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2$ (Lemma B.22). On the other hand, it follows from Proposition 6.8 that $\mathbf{P}_n^\circ(S)$ is generated by $\mathbf{P}_n$ under the action of $\mathbf{B}_n(S)$. As a consequence, the map $\mathbf{P}_n^{\mathrm{ab}} \to (\mathbf{P}_n^\circ(S)^{\mathrm{ab}})_{\mathbf{B}_n(S)}$ induced by φ is surjective. Moreover, it factors through $(\mathbf{P}_n^{\mathrm{ab}})_{\mathbf{B}_n} = (\mathbf{P}_n^{\mathrm{ab}})_{\mathfrak{S}_n} \cong \mathbb{Z}$. Thus $(\mathbf{P}_n^\circ(S)^{\mathrm{ab}})_{\mathbf{B}_n(S)}$ is cyclic, and its image in $\mathbf{B}_n(S)^{\mathrm{ab}}$ is generated by σ^2, which is the image of any pure braid generator.

All this implies that $\mathbf{B}_n(S)^{\mathrm{ab}}/\langle \sigma \rangle \cong \pi_1(S)^{\mathrm{ab}}$. Moreover, the corresponding projection $\mathbf{B}_n(S)^{\mathrm{ab}} \twoheadrightarrow \pi_1(S)^{\mathrm{ab}}$ splits, a splitting being induced by any of the ψ_i. As a consequence, $\mathbf{B}_n(S)^{\mathrm{ab}}$ identifies with $\pi_1(S)^{\mathrm{ab}} \times \langle \sigma \rangle$, and we can use Proposition 6.10 to get a complete calculation. $\qquad\square$

6.3.2 The lower central series

Let us now turn to the study of the lower central series of $\mathbf{B}_n(S)$. Our main tool for studying it is the decomposition theorem 6.13 below, whose proof relies on the following:

Lemma 6.12 *Let $n \geqslant 3$. The image of $\mathbf{B}_n$ in $\mathbf{B}_n(S)/\Gamma_\infty$ is cyclic, and it is central. Namely, it is generated by the common class σ of the usual generators σ_i of $\mathbf{B}_n$.*

Proof. The morphism φ sends $\Gamma_\infty(\mathbf{B}_n)$ to $\Gamma_\infty(\mathbf{B}_n(S))$. From $\sigma_i\sigma_j^{-1} \in \Gamma_\infty(\mathbf{B}_n)$ for all $i, j < n$, we deduce that $\sigma_i \equiv \sigma_j \pmod{\Gamma_\infty(\mathbf{B}_n(S))}$. Let us denote by $\sigma \in \mathbf{B}_n(S)/\Gamma_\infty$ the common image of the σ_i's. Since the σ_i's generate $\mathbf{B}_n$, the image of $\mathbf{B}_n$ in $\mathbf{B}_n(S)/\Gamma_\infty$ is the cyclic subgroup generated by σ. In particular, its elements commute with σ. Moreover, for all $\gamma \in \pi_1(S - \mathbb{D}^\mathrm{o})$, the braids σ_2 and $\psi_1(\gamma)$ have disjoint support, hence σ also commutes with the image of ψ_1. Since the images of φ and ψ_1 generate $\mathbf{B}_n(S)$ by Proposition 6.8, this means that σ is a central element of $\mathbf{B}_n(S)/\Gamma_\infty$. $\qquad\square$

We can now state our main decomposition theorem:

Theorem 6.13 *For all $n \geqslant 3$, there is a central extension:*

$$\langle \sigma^2 \rangle \lhook\joinrel\longrightarrow \mathbf{B}_n(S)/\Gamma_\infty \longrightarrow\!\!\!\!\to \pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2.$$

Proof. Since π_S is surjective, it sends $\Gamma_\infty(\mathbf{B}_n(S))$ onto a normal subgroup of $\pi_1(S) \wr \mathfrak{S}_n$. This normal subgroup is contained in $\Gamma_\infty(\pi_1(S) \wr \mathfrak{S}_n)$ and contains the $\tau_i\tau_j^{-1}$, which are the images of the $\sigma_i\sigma_j^{-1} \in \Gamma_\infty(\mathbf{B}_n(S))$. By Lemma B.24, it is equal to $\Gamma_2(\pi_1(S) \wr \mathfrak{S}_n)$. Then, we can apply the Nine Lemma to the diagram:

$$
\begin{array}{ccccc}
\Gamma_\infty(\mathbf{B}_n(S)) \cap \mathbf{P}_n^\circ(S) & \lhook\joinrel\longrightarrow & \Gamma_\infty(\mathbf{B}_n(S)) & \longrightarrow\!\!\!\!\to & \Gamma_2(\pi_1(S) \wr \mathfrak{S}_n). \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{P}_n^\circ(S) & \lhook\joinrel\longrightarrow & \mathbf{B}_n(S) & \xrightarrow{\ \pi_S\ } & \pi_1(S) \wr \mathfrak{S}_n \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{P}_n^\circ(S)/\Gamma_\infty(\mathbf{B}_n(S)) \cap \mathbf{P}_n^\circ(S) & \dashrightarrow & \mathbf{B}_n(S)/\Gamma_\infty & \overset{\overline{\pi}_S}{\dashrightarrow} & (\pi_1(S) \wr \mathfrak{S}_n)^{\mathrm{ab}}.
\end{array}
$$

Then recall that $(\pi_1(S) \wr \mathfrak{S}_n)^{\mathrm{ab}} \cong \pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2$. Thus, we are left with analysing the kernel $\mathbf{P}_n^\circ(S)/\Gamma_\infty(\mathbf{B}_n(S)) \cap \mathbf{P}_n^\circ(S)$ of $\overline{\pi}_S$, which is the image of $\mathbf{P}_n^\circ(S)$ in $\mathbf{B}_n(S)/\Gamma_\infty$. Since $\mathbf{P}_n^\circ(S)$ is the normal closure of $\mathbf{P}_n$ in $\mathbf{B}_n(S)$ by Proposition 6.8, its image in $\mathbf{B}_n(S)/\Gamma_\infty$ is the normal closure of the image of $\mathbf{P}_n$. But $\mathbf{P}_n$ is sent to $\langle \sigma^2 \rangle$ which is central (and, in particular, normal) in $\mathbf{B}_n(S)/\Gamma_\infty$, whence the result. $\qquad\square$

Remark 6.14 We also have a central extension:

$$\langle \sigma \rangle \lhook\joinrel\longrightarrow \mathbf{B}_n(S)/\Gamma_\infty \longrightarrow\!\!\!\!\to \pi_1(S)^{\mathrm{ab}}.$$

This slightly different statement tells us slightly different things: it implies that σ is central, whereas the statement of the theorem says that $\overline{\sigma}$ is not trivial in $\mathbf{B}_n(S)^{\mathrm{ab}}$.

Corollary 6.15 *For any $n \geqslant 3$, we have $\Gamma_3(\mathbf{B}_n(S)) = \Gamma_4(\mathbf{B}_n(S))$.*

Proof. Proposition 6.13 implies that $\mathbf{B}_n(S)/\Gamma_\infty$ is 2-nilpotent, which means exactly that its Γ_3 is trivial. In other words, $\Gamma_3 \subseteq \Gamma_\infty$ for $\mathbf{B}_n(S)$. $\qquad\square$

The remaining cases consist in the cases when $n = 1$ and when $n = 2$.

Proposition 6.16 *For any connected surface S, either $\mathbf{B}_1(S) = \pi_1(S)$ is abelian, which occurs precisely when $S \in \{\mathbb{D} - pt, \mathbb{D}, \mathbb{S}^2, \mathbb{T}^2, \mathbb{P}^2, \mathbb{M}^2\}$ up to isotopy equivalence, or its lower central series does not stop.*

Proof. If S is closed and not in $\{\mathbb{S}^2, \mathbb{P}^2, \mathbb{T}^2\}$ then the lower central series of $\pi_1(S)$ does not stop. To see this, note that it admits a presentation of the form $\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$ or $\langle c_1, \ldots, c_g \mid c_1^2 \cdots c_g^2 = 1 \rangle$ for $g \geqslant 2$, and these both project onto $\mathbb{Z}/2 * \mathbb{Z}/2 = \langle x, y \mid x^2 = y^2 = 1 \rangle$ by sending a_1, b_1, c_1 to x and all other generators to y; the lower central series of $\mathbb{Z}/2 * \mathbb{Z}/2$ does not stop by Proposition B.16. If S is non-compact, its fundamental group is free by Proposition 6.4. If it is also not in $\{\mathbb{D} - pt, \mathbb{D}, \mathbb{M}^2\}$ then its fundamental group is moreover *non-abelian* free, and hence its lower central series does not stop by [Mag35]. $\qquad\square$

Proposition 6.17 *When S is not $\mathbb{D}$, $\mathbb{S}^2$ or $\mathbb{P}^2$, up to isotopy equivalence, the lower central series of $\mathbf{B}_2(S)$ does not stop.*

Proof. The group $\mathbf{B}_2(S)$ surjects onto $\pi_1(S) \wr \mathfrak{S}_2$. Since $\pi_1(S)^{\mathrm{ab}}$ surjects onto $\mathbb{Z}$ except in the three excluded cases of the statement, we can apply Corollary B.27 to see that the lower central series of $\pi_1(S) \wr \mathfrak{S}_2$ does not stop. Then the one of $\mathbf{B}_n(S)$ does not either by Lemma 1.1. $\qquad\square$

Quote[BB09b].

Remark 6.18 Clearly, the lower central series of $\mathbf{B}_2(\mathbb{D}) \cong \mathbb{Z}$ and of $\mathbf{B}_2(\mathbb{S}^2) \cong \mathbb{Z}/2$ both stop at Γ_2. The fact that $\mathbf{B}_2(\mathbb{S}^2) \cong \mathbb{Z}/2$ is part of Fadell and Van Buskirk's Theorem quoted as Corollary A.18. On the other hand, the group $\mathbf{B}_2(\mathbb{P}^2)$ is the dicyclic group of order 16 [Bus66, p. 87], which is 3-nilpotent, so its lower central series stops at Γ_4.

6.3.3 The Lie ring

We can be more precise about our description of the lower central series of surface braid groups. In particular, Corollary 6.15 says that for $n \geqslant 3$, the lower central series stops at most at Γ_3, but it does not say when it stops at Γ_2 (then the associated Lie ring consists only of the abelianisation, which has already been computed in Proposition 6.11), or when it stops at Γ_3 (in which case the Lie ring is 2-nilpotent but not abelian). We now show that the latter holds only for non-planar orientable surfaces, and we compute precisely $\mathcal{L}(\mathbf{B}_n(S))$, generalising results of [BGG08].

Theorem 6.19 *Let $n \geqslant 3$ be an integer and S be a connected surface. The lower central series of $\mathbf{B}_n(S)$:*

- *stops at Γ_2 if S is either planar or non-orientable, or if $S \cong \mathbb{S}^2$.*
- *stops at Γ_3 in the other cases. Then $\mathcal{L}_2(\mathbf{B}_n(S))$ is cyclic, generated by the common class σ^2 of the pure braid generators A_{ij}.*

Moreover, in the second case, σ^2 is of finite order if and only if S is closed, in which case its order is $n + g - 1$, where g is the genus of S.

Proof. Since the lower central series of $\mathbf{B}_n(S)$ stops at most at Γ_3, the subquotient $\mathcal{L}_2(\mathbf{B}_n(S))$ identifies with $\Gamma_2(\mathbf{B}_n(S)/\Gamma_\infty)$ (the latter being $\Gamma_2/\Gamma_\infty(\mathbf{B}_n(S)) = \Gamma_2/\Gamma_3(\mathbf{B}_n(S))$). Moreover, using the central extension of Theorem 6.13, we see that in $\mathbf{B}_n(S)/\Gamma_\infty$, the subgroup Γ_2 must be contained in $\langle \sigma^2 \rangle$, which implies that it is cyclic, generated by a power of σ^2.

Planar surfaces. If S is planar, then the common class σ of the σ_i in $\mathbf{B}_n(S)^{\mathrm{ab}} = (\mathbf{B}_n(S)/\Gamma_\infty)^{\mathrm{ab}}$ is of infinite order by Proposition 6.11. As a consequence, $\Gamma_2(\mathbf{B}_n(S)/\Gamma_\infty)$ does not contain any power of $\sigma = \overline{\sigma_1}$. But $\Gamma_2(\mathbf{B}_n(S)/\Gamma_\infty)$ is contained in $\langle \sigma^2 \rangle$, so it must be trivial, which means that $\Gamma_2 = \Gamma_\infty$ for $\mathbf{B}_n(S)$.

The sphere. If $S = \mathbb{S}^2$, then $\mathbf{B}_n(\mathbb{S}^2)$ is a quotient of $\mathbf{B}_n$ (see Corollary A.18), which implies that its lower central series also stops at Γ_2.

Non-orientable surfaces. The surface S is non-orientable if and only if it contains an embedded Möbius band. Then $\sigma_1^2 = [\sigma_1 c \sigma_1^{-1}, c^{-1}]$ for some $c \in \mathbf{B}_n(S)$: precisely, c is a braid whose first strand goes around the Möbius strip once, that is, through the crosscap; see Figure 6.1. Since σ_1 is sent to the central element σ of $\mathbf{B}_n(S)/\Gamma_\infty$, this relation implies that $\sigma^2 = [\overline{c}, \overline{c}^{-1}] = 1$. Thus $\langle \sigma^2 \rangle$ is trivial, and so is its subgroup $\Gamma_2(\mathbf{B}_n(S)/\Gamma_\infty) \cong \mathcal{L}_2(\mathbf{B}_n(S))$.

Non-planar orientable surfaces. If S has a handle, then $\sigma_1^2 = [a^{-1}, \sigma_1 b \sigma_1^{-1}]$ for some $a, b \in \mathbf{B}_n(S)$. Precisely, a and b are braids whose first strands go around a handle; see Figure 6.2. Hence $\sigma^2 \in \Gamma_2(\mathbf{B}_n(S)/\Gamma_\infty)$, which implies that $\mathcal{L}_2(\mathbf{B}_n(S))$ is generated by σ^2.

If S is a **non-planar compact orientable surface with at least one boundary component**, then S can be embedded in some $\Sigma_{g,1}$ for an arbitrary large g, by attaching a disc to each boundary component save one. The induced map $\mathcal{L}_2(\mathbf{B}_n(S)) \to \mathcal{L}_2(\mathbf{B}_n(\Sigma_{g,1}))$ sends σ^2 to σ^2, and the latter is of infinite order (Proposition 6.20).

If S is a **non-compact non-planar orientable surface**, let us suppose that σ^2 is a torsion element in $\mathbf{B}_n(S)/\Gamma_\infty = \mathbf{B}_n(S)/\Gamma_3$. Then for some integer k, σ_1^{2k} is equal to some product of commutators of length at least 3 in $\mathbf{B}_n(S)$. Such a formula involves only a finite number of braids. Let us choose a representative of each of these isotopy classes. These involve a finite number of paths on the surface. If moreover we choose an isotopy realising the aforementioned equality of braids (using the concatenation of the chosen representatives as a representative of the right-hand side), the image of this isotopy is contained in a compact subsurface S' of S. Thus, our formula also holds in $\mathbf{B}_n(S')$: σ_1^{2k} is equal to some product of commutators of length at least 3 in there. Then $\sigma^{2k} = 1$ in $\mathbf{B}_n(S')/\Gamma_3$. However, this contradicts the previous case, since S' cannot be closed (nor planar). We conclude that σ^2 cannot be a torsion element in $\mathbf{B}_n(S)/\Gamma_\infty$; equivalently, $\mathcal{L}_2(\mathbf{B}_n(S)) \cong \mathbb{Z}$.

If S is a **closed orientable surface** Σ_g of genus $g \geqslant 1$, then $\mathcal{L}_2(\mathbf{B}_n(\Sigma_g)) \cong \mathbb{Z}/(n+g-1)$. This is [BGG08, Th. 1], which can be deduced from Proposition 6.20 below. We will do so in a more general context later; see Proposition 6.34. $\qquad\square$

The proof of the following result is inspired from [BGG08]. In fact, it is equivalent to [BGG08, Th. 1] in a quite straightforward way; see Remark 6.21.

Proposition 6.20 *Let $g \geqslant 1$, and let $\Sigma_{g,1}$ denote the compact surface of genus g with one boundary component. For every $n \geqslant 1$, $\mathcal{L}_2(\mathbf{B}_n(\Sigma_{g,1})) \cong \mathbb{Z}$, generated by the common class σ^2 of the pure braid generators.*

Proof. We compute completely $\mathbf{B}_n(\Sigma_{g,1})/\Gamma_3$. Let us consider the quotient of $\mathbf{B}_n(\Sigma_{g,1})$ by the relations $\sigma_i = \sigma_{i+1}$, that is, by the normal closure N of the $\sigma_i \sigma_{i+1}^{-1}$ for $1 \leqslant i \leqslant n-1$. We already know that, modulo Γ_3, the braid generators σ_i have a common class σ. In particular, $N \subseteq \Gamma_3$, and we will show that $N = \Gamma_3$ by showing that $G := \mathbf{B}_n(\Sigma_{g,1})/N$ is 2-nilpotent. Thanks to Proposition A.14, we can compute a presentation of this quotient. It is generated by σ, together with a_r and b_r for $1 \leqslant r \leqslant g$. The braid relations on the σ_i become trivial there. $(BS1)$ says that σ is central. $(BS2)$ says that the a_r and the b_s commute with one another, except a_r and b_r (for each r). Since σ is central, $(BS3)$ becomes trivial. Finally, $(BS4)$ can be written as $b_r a_r b_r^{-1} = a_r \sigma^{-2}$. From this presentation, one can see that $G \cong (\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$, where the three factors are free abelian on σ, the a_r and the b_r respectively; the action of each b_r is trivial on σ and the a_s if $s \neq r$, and $b_r \cdot a_r = a_r - 2\sigma$. Since this group is 2-nilpotent, we have $N = \Gamma_3(\mathbf{B}_n(\Sigma_{g,1}))$, as announced. Moreover, the Lie ring of G, which is the Lie ring of $\mathbf{B}_n(\Sigma_{g,1})$ by Corollary 6.15, is easy to compute. Namely, $\mathcal{L}_1(G) = G^{\mathrm{ab}} \cong (\mathbb{Z}/2)^2 \times \mathbb{Z}^{2g}$, $\mathcal{L}_2(G) = \Gamma_2(G) \cong \mathbb{Z}$ is generated by $\overline{\sigma^2}$, and the only non-trivial brackets of generators are $[\overline{a}_r, \overline{b}_r] = \overline{\sigma^2}$. $\qquad\square$

Remark 6.21 Here we choose to recover [BGG08, Th. 1] from Proposition 6.20. However, one can also *deduce* Proposition 6.20 from [BGG08, Th. 1]. Namely, one can embed $\Sigma_{g,1}$ into $\Sigma_{g'}$ for any $g' \geqslant g$, by attaching $\Sigma_{g'-g,1}$ along the boundary component. The induced map $\mathcal{L}_2(\mathbf{B}_n(\Sigma_{g,1})) \to \mathcal{L}_2(\mathbf{B}_n(\Sigma_{g'}))$ sends σ^2 to σ^2, and the latter is of order $n + g' - 1$ by [BGG08, Th. 1]. Thus, by varying g', we see that σ^2 cannot be of finite order inside $\mathcal{L}_2(\mathbf{B}_n(\Sigma_{g,1}))$.

Remark 6.22 The group $(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$ appearing in the proof of Proposition 6.20, which is then the maximal nilpotent quotient of $\mathbf{B}_n(\Sigma_{g,1})$, has a nice interpretation as a matrix group resembling the Heisenberg group of a symplectic vector space. Namely, it is the subgroup of $\mathrm{GL}_{g+2}(\mathbb{Q})$ given

by:

$$\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} & \frac{1}{2}\mathbb{Z} \\ & 1 & 0 & \cdots & 0 & \mathbb{Z} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & 0 & \mathbb{Z} \\ & & & & 1 & \mathbb{Z} \\ & & & & & 1 \end{pmatrix}$$

Remark 6.23 (About the Riemann-Hurwitz formula) The result from [BGG08, Th. 1] quoted above and recovered below says that when S is an orientable closed surface, the information encoded in $\mathcal{L}_2(\mathbf{B}_n(S))$ is essentially the genus of S, or its Euler characteristic. In fact, one can recover the Riemann-Hurwitz formula for coverings of closed surfaces from this computation. Indeed, let $p\colon \Sigma_h \twoheadrightarrow \Sigma_g$ be a k-covering. It induces a continuous map $p^*\colon C_n(\Sigma_g) \to C_{kn}(\Sigma_h)$ between unordered configuration spaces sending a configuration of n points to the configuration of the kn preimages of these points by p. The induced map between the fundamental groups sends the pure braid generator A_{12} to the product of the $A_{j,j+1}$ for $j \in p^{-1}(1)$ (if the correct conventions are chosen). Thus the induced map from $\mathcal{L}_2(\mathbf{B}_n(\Sigma_g))$ to $\mathcal{L}_2(\mathbf{B}_{kn}(\Sigma_h))$ sends σ^2 to $k\sigma^2$. Since σ^2 is of order $n + g - 1$ in $\mathcal{L}_2(\mathbf{B}_n(\Sigma_g))$, we obtain $(n + g - 1)k\sigma^2 = 0$ in $\mathcal{L}_2(\mathbf{B}_{kn}(\Sigma_h))$. However, σ^2 is of order $kn + h - 1$ there, implying that $kn + h - 1$ divides $kn + k(g - 1)$. This holds for all n. For n big enough, $kn + k(g-1) < 2(kn + h - 1)$, so the only possibility is that $kn + k(g-1) = kn + h - 1$, that is, $k(g - 1) = h - 1$, as announced.

6.4 Partitioned braids on surfaces

Let us now study the lower central series of partitioned surface braid groups $\mathbf{B}_\lambda(S)$, generalising the results from §6.3, which can be seen as the case of the trivial partition $\lambda = (n)$ of n. We follow the same steps: we first compute the abelianisations of $\mathbf{B}_\lambda(S)$ in Proposition 6.26, before studying $\mathbf{B}_\lambda(S)/\Gamma_\infty$ and showing that the lower central series of $\mathbf{B}_\lambda(S)$ stops at most at Γ_3 when the partition λ has only blocks of size at least 3 in Theorem 6.28 and Corollary 6.29. Finally, we compute the associated Lie rings in Theorem 6.30.

Definition 6.24 Let S be a surface, let $n \geqslant 1$ be an integer, and let $\lambda = (n_1, ..., n_l)$ be a partition of n. The corresponding *partitioned surface braid group* is:

$$\mathbf{B}_\lambda(S) := \pi^{-1}(\mathfrak{S}_\lambda) = \pi^{-1}\left(\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}\right) \subseteq \mathbf{B}_n(S).$$

There are canonical surjections between partitioned braid groups, obtained by forgetting blocks. For most surfaces, these projections behave exactly as they do for the disc. However, their behaviour for closed surfaces is somewhat trickier, especially when it comes to the sphere and the projective plane:

Proposition 6.25 (Fadell-Neuwirth exact sequences) *Let S be a connected surface, let μ be a partition of an integer $m \geqslant 1$, ν be a partition of an integer $n \geqslant 1$, and let us denote by $\mu\nu$ their concatenation, which is a partition of $m + n$. The following sequence of canonical maps is exact:*

$$\mathbf{B}_\mu(S - \{n \ pts\}) \longrightarrow \mathbf{B}_{\mu\nu}(S) \longrightarrow \mathbf{B}_\nu(S) \longrightarrow 1.$$

Moreover, except when $S = \mathbb{S}^2$ and $n = 1, 2$ or $S = \mathbb{P}^2$ and $n = 1$, this is in fact a short exact sequence:

$$1 \longrightarrow \mathbf{B}_\mu(S - \{n \ pts\}) \longrightarrow \mathbf{B}_{\mu\nu}(S) \longrightarrow \mathbf{B}_\nu(S) \longrightarrow 1.$$

Furthermore, if S is not closed, the surjection $\mathbf{B}_{\mu\nu}(S) \twoheadrightarrow \mathbf{B}_\nu(S)$ splits.

Proof. Recall that if λ is a partition of N, we denote by $C_\lambda(S)$ the configuration space $F_N(S)/\mathfrak{S}_\lambda$. Forgetting the first n points induces a map of configuration spaces $C_{\mu\nu}(S) \to C_\nu(S)$, which is a

locally trivial fibration with fibres homeomorphic to $C_\mu(S - \{n\ pts\})$ [FN62, Th. 3]. Since its fibres are path-connected, part of its long exact sequence of homotopy groups is:

$$\pi_2(C_\nu(S)) \to \mathbf{B}_\mu(S - \{n\ pts\}) \longrightarrow \mathbf{B}_{\mu\nu}(S) \longrightarrow \mathbf{B}_\nu(S) \longrightarrow 1.$$

The map $F_n(S) \to C_\nu(S)$ is a covering, so that $\pi_2(C_\nu(S)) \cong \pi_2(F_n(S))$ is trivial except when $S = \mathbb{S}^2$ and $n = 1, 2$ or $S = \mathbb{P}^2$ and $n = 1$. More precisely, when $S = \mathbb{S}^2$ this is [FV62, Cor. p. 244], when $S = \mathbb{P}^2$ it is [Bus66, Cor. p. 82] and in all other cases it is [FN62, Cor. 2.2].

If S is not closed, then there is an isotopy equivalence between S and a proper subsurface S' of S. Then one can choose a configuration of m points in $S - S'$ and add them to each configuration of n points of S', getting a map $F_n(S') \to F_{m+n}(S)$. The induced map $\pi_1(C_\nu(S)) \cong \pi_1(C_\nu(S')) \to \pi_1(C_{\mu\nu}(S))$ is the required section. $\qquad\square$

6.4.1 The abelianisation

Let $\lambda = (n_1, ..., n_l)$ be a partition of an integer n. A computation of $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ can be obtained by a quite straightforward generalisation of the computation of $\mathbf{B}_n(S)^{\mathrm{ab}}$ from Proposition 6.11.

Let us first recall that the morphism φ from §6.2 induces a map $\mathbf{B}_\lambda^{\mathrm{ab}} \to \mathbf{B}_\lambda(S)^{\mathrm{ab}}$. Then, from Proposition 3.4, we get that:

- For each $i \leqslant l$ such that $n_i \geqslant 2$, all the σ_α with α and $\alpha + 1$ in the i-th block of λ have a common image in $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$, called s_i.
- For each $i < j \leqslant l$, all the $A_{\alpha\beta}$ with α (resp. β) in the i-th (resp. the j-th) block of λ have a common image in $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$, called a_{ij} (or sometimes a_{ji}).

Let us now consider the short exact sequence:

$$\mathbf{P}_n^\circ(S) \lhook\joinrel\longrightarrow \mathbf{B}_\lambda(S) \xrightarrow{\ \pi_S\ }\!\!\!\!\twoheadrightarrow \pi_1(S) \wr \mathfrak{S}_\lambda.$$

We can apply Lemma 1.8 to it, and we get an exact sequence of abelian groups:

$$(\mathbf{P}_n^\circ(S)^{\mathrm{ab}})_{\mathbf{B}_\lambda(S)} \longrightarrow \mathbf{B}_\lambda(S)^{\mathrm{ab}} \longrightarrow (\pi_1(S) \wr \mathfrak{S}_\lambda)^{\mathrm{ab}} \longrightarrow 0.$$

On the one hand, the quotient $(\pi_1(S) \wr \mathfrak{S}_\lambda)^{\mathrm{ab}}$ is isomorphic to the product of the $(\pi_1(S) \wr \mathfrak{S}_{n_i})^{\mathrm{ab}}$, which is $(\pi_1(S)^{\mathrm{ab}})^l \times (\mathbb{Z}/2)^{l'}$ by Corollary B.23 where l' is the number of indices $i \leqslant l$ such that $n_i \geqslant 2$. On the other hand, it follows from Proposition 6.8 that $\mathbf{P}_n^\circ(S)$ is generated by $\mathbf{P}_n$ under the action of $\mathbf{P}_n(S)$, which is a subgroup of $\mathbf{B}_\lambda(S)$. As a consequence, the map $\mathbf{P}_n^{\mathrm{ab}} \to (\mathbf{P}_n^\circ(S)^{\mathrm{ab}})_{\mathbf{B}_\lambda(S)}$ induced by φ is surjective. Moreover, it factors through the quotient $(\mathbf{P}_n^{\mathrm{ab}})_{\mathbf{B}_\lambda} = (\mathbf{P}_n^{\mathrm{ab}})_{\mathfrak{S}_\lambda} \cong \mathbb{Z}^{l(l-1)/2} \times \mathbb{Z}^{l'}$. Thus the image of $(\mathbf{P}_n^\circ(S)^{\mathrm{ab}})_{\mathbf{B}_\lambda(S)}$ in $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ is generated by the images of the pure braid generators, which are the elements $2s_i$ for $i \leqslant l$ such that $n_i \geqslant 2$, and a_{ij} for $i < j \leqslant l$.

Now, let H be the subgroup of $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ generated by the s_i and the a_{ij}. From the above, we get an isomorphism $\mathbf{B}_\lambda(S)^{\mathrm{ab}}/H \cong (\pi_1(S)^{\mathrm{ab}})^l$. Moreover, the corresponding projection $\mathbf{B}_\lambda(S)^{\mathrm{ab}} \twoheadrightarrow (\pi_1(S)^{\mathrm{ab}})^l$ splits: a splitting is induced by the product of the ψ_i, where each ψ_i is induced by any ψ_α, for α in the i-th block of λ. As a consequence, $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ identifies with $(\pi_1(S)^{\mathrm{ab}})^l \times H$, and we can use Proposition 6.10 to get a complete calculation:

Proposition 6.26 *Let $n \geqslant 1$ and $\lambda = (n_1, ..., n_l)$ be a partition of n. Then:*

$$\mathbf{B}_\lambda(S)^{\mathrm{ab}} \cong \left(\pi_1(S)^{\mathrm{ab}}\right)^l \times \left(\mathbf{B}_\lambda^{\mathrm{ab}}/R\right),$$

where $\mathbf{B}_\lambda^{\mathrm{ab}}$ is free abelian on the s_i and the a_{ij} from Proposition 3.4. Moreover, R is:

- *trivial if S is planar,*
- *generated by the relations $2(n_i - 1)s_i + \sum_{j \neq i} n_j a_{ij}$ (for all $1 \leqslant i \leqslant n$) if $S \cong \mathbb{S}^2$,*
- *generated by the $2s_i$ and the a_{ij} in all the other cases.*

Explicitly, if l' denotes the number of indices $i \leqslant l$ such that $n_i \geqslant 2$, we have, for $S \not\cong \mathbb{S}^2$:

$$\mathbf{B}_\lambda^{\mathrm{ab}}/R \;\cong\; \begin{cases} \mathbb{Z}^{l'} \times \mathbb{Z}^{l(l-1)/2} & \text{if } S \text{ is planar,} \\ (\mathbb{Z}/2)^{l'} & \text{if } S \text{ does not embed into the sphere.} \end{cases}$$

Proof. If $S \neq \mathbb{S}^2$, all the relations are direct consequences of Proposition 6.10, and it remains to prove that they are the only ones. In the first case, we can directly use the result of Proposition 3.4: an embedding of S into $\mathbb{R}^2 \cong \mathbb{S}^2 - pt$ induces a retraction $\mathbf{B}_\lambda(S)^{\mathrm{ab}} \twoheadrightarrow \mathbf{B}_\lambda^{\mathrm{ab}}$ of the morphism $\mathbf{B}_\lambda^{\mathrm{ab}} \to \mathbf{B}_\lambda(S)^{\mathrm{ab}}$ induced by φ, whence the result in this case. In the third one, the same proof as the proof of Proposition 3.4 works: projections onto the factors are given by projections onto the $\mathfrak{S}_{n_i}^{\mathrm{ab}} \cong \mathbb{Z}/2$ for $n_i \geqslant 2$.

If $S = \mathbb{S}^2$, Corollary A.22 describes $\mathbf{B}_\lambda(\mathbb{S}^2)$ as the quotient of $\mathbf{B}_\lambda$ by the relations $A_{\alpha 1} A_{\alpha 2} \cdots A_{\alpha n} = 1$ for all $1 \leqslant \alpha \leqslant n$. The abelianisation $\mathbf{B}_\lambda(\mathbb{S}^2)^{\mathrm{ab}}$ is then the quotient of $\mathbf{B}_\lambda^{\mathrm{ab}}$ (described in Proposition 3.4) by the classes of the above relations. If α is in the i-th block of λ, then the class of $A_{\alpha\beta}$ is either $2s_i$ if β is also in the i-th block (which holds for $n_i - 1$ values of β), or a_{ij} if β is in the j-th block for some $j \neq i$ (which happens for n_j values of β, for each $j \neq i$). Thus the classes of the $A_{\alpha 1} A_{\alpha 2} \cdots A_{\alpha n}$ are indeed the relations of the statement. $\square$

6.4.2 The lower central series

The work done above for showing that the lower central series of $\mathbf{B}_n(S)$ stops if $n \geqslant 3$ (see Corollary 6.15) generalises to the partitioned braid group when all the blocks of the partition are of size at least 3. First, here follows the generalisation of Lemma 6.12.

Lemma 6.27 *Let $\lambda = (n_1, ..., n_l)$ be a partition of n, with $n_i \geqslant 3$ for all i. The image of $\mathbf{B}_\lambda$ in $\mathbf{B}_\lambda(S)/\Gamma_\infty$ is central. In particular, it is a quotient of $\mathbf{B}_\lambda^{\mathrm{ab}}$.*

Proof. It follows from Theorem 3.5 that $\Gamma_\infty(\mathbf{B}_\lambda) = \Gamma_2(\mathbf{B}_\lambda)$. As a consequence, the morphism $\varphi : \mathbf{B}_\lambda \to \mathbf{B}_\lambda(S)/\Gamma_\infty$ factors through $\mathbf{B}_\lambda/\Gamma_2 = \mathbf{B}_\lambda^{\mathrm{ab}}$. Hence its image is abelian, generated by the images of the s_i and the a_{ij} from Proposition 3.4.

In order to show that it is central, we need to show that these elements commute with generators of $\mathbf{B}_\lambda(S)/\Gamma_\infty$. We deduce from Proposition 6.8 that $\mathbf{B}_\lambda(S)$ is generated by the images of φ and of the ψ_α. In fact, we can restrict to taking one α in each block of λ, since ψ_α and ψ_β are conjugated by elements of $\mathbf{B}_\lambda$ if α and β are in the same block. The s_i and the a_{ij} already commute with each other, so we only need to show that they commute with the images of the ψ_α. Since all $n_i \geqslant 3$, we can find representatives of all the s_i and the a_{ij} whose support is disjoint from elements of $\mathrm{Im}(\psi_\alpha)$ for each fixed α. Thus, the s_i and the a_{ij} commute with a family of generators of $\mathbf{B}_\lambda(S)/\Gamma_\infty$, which proves our claim. $\square$

We can now generalise our main decomposition theorem (Theorem 6.13) to partitioned braids:

Theorem 6.28 *Let $\lambda = (n_1, ..., n_l)$ be a partition of n, with $n_i \geqslant 3$ for all i. There is a central extension:*

$$\langle s_i^2, a_{ij} \rangle_{i,j \leqslant l} \;\lhook\joinrel\longrightarrow\; \mathbf{B}_\lambda(S)/\Gamma_\infty \;\longrightarrow\joinrel\twoheadrightarrow\; (\pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2)^l.$$

Proof. The proof is essentially the same as the proof of Theorem 6.13, so we only stress what changes. The element $\sigma_\alpha \sigma_\beta^{-1}$ is in $\Gamma_\infty(\mathbf{B}_n(S))$ only when α and β are in the same block of λ. However, their images $\tau_\alpha \tau_\beta^{-1}$ in $\pi_1(S) \wr \mathfrak{S}_\lambda$ still normally generate $\Gamma_2(\pi_1(S) \wr \mathfrak{S}_\lambda)$, because $\pi_1(S) \wr \mathfrak{S}_\lambda$ is the product of the $\pi_1(S) \wr \mathfrak{S}_{n_i}$, whose Γ_2 is normally generated by the $\tau_\alpha \tau_\beta^{-1}$ for α and β are in the i-th block of λ (Lemma B.24). Thus, arguing exactly as in the proof of Theorem 6.13, we get a short exact sequence:

$$\mathbf{P}_n^\circ(S)/\Gamma_\infty(\mathbf{B}_\lambda(S)) \cap \mathbf{P}_n^\circ(S) \;\lhook\joinrel\longrightarrow\; \mathbf{B}_\lambda(S)/\Gamma_\infty \;\xrightarrow{\;\pi_S\;}\joinrel\twoheadrightarrow\; (\pi_1(S) \wr \mathfrak{S}_\lambda)^{\mathrm{ab}}.$$

We recall that $(\pi_1(S) \wr \mathfrak{S}_\lambda)^{\mathrm{ab}} \cong (\pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2)^l$. Moreover, the kernel is the normal closure of the image of $\mathbf{P}_n$, but this image is already normal, and even central by Lemma 6.27, generated by the a_{ij} and the squares of the s_i. $\qquad\square$

Finally, Corollary 6.15 adapts readily to this context:

Corollary 6.29 *If all the blocks of λ have size at least 3, then $\Gamma_3(\mathbf{B}_\lambda(S)) = \Gamma_4(\mathbf{B}_\lambda(S))$.*

6.4.3 The Lie ring

Here again, in the stable case (that is, when all the blocks of our partition λ have size at least 3), we can be more precise about the description of the lower central series. Namely, we generalise the calculations of Theorem 6.19 as follows:

Theorem 6.30 *Let $\lambda = (n_1, ..., n_l)$ be a partition of an integer n with $n_i \geqslant 3$ for all i, and let S be a connected surface. The lower central series of $\mathbf{B}_\lambda(S)$:*

- *stops at Γ_2 if S is planar, if $S \cong \mathbb{S}^2$ or if $l = 1$ and S is non-orientable.*
- *stops at Γ_3 in the other cases.*

Remark 6.31 In both cases, the Lie ring can be computed completely. Namely, $\mathcal{L}_1(\mathbf{B}_\lambda(S)) = \mathbf{B}_\lambda(S)^{\mathrm{ab}}$ has already been computed in Proposition 6.26. In the first case, no further computation is required. In the second case, $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ is described completely in Proposition 6.33 and Proposition 6.34 for orientable surfaces, and in Corollary 6.37 and Corollary 6.39 for non-orientable ones. Moreover, one can easily describe the Lie bracket from the computations given there. Precisely, the only non-trivial brackets are coming from the computations depicted in Figures 6.2 and 6.1.

Proof of Theorem 6.30. Let $\lambda = (n_1, ..., n_l)$ be a partition of n with all the blocks of size at least 3. We know that $\Gamma_3 = \Gamma_\infty$ for $\mathbf{B}_\lambda(S)$ (see Corollary 6.29). Then $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ identifies with $\Gamma_2(\mathbf{B}_\lambda(S)/\Gamma_\infty)$. Moreover, in $\mathbf{B}_\lambda(S)/\Gamma_\infty$, the subgroup Γ_2 must be contained in the kernel $\langle s_i^2, a_{ij} \rangle_{i,j \leqslant l}$ of the central extension of Theorem 6.28.

Planar surfaces. If S is planar, then the canonical projection from $\mathbf{B}_\lambda(S)/\Gamma_\infty$ to $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$ sends the s_i and the a_{ij} to their counterparts in $\mathbf{B}_\lambda(S)^{\mathrm{ab}}$. The latter is a linearly independent family by Proposition 6.26, so the restriction $\langle s_i^2, a_{ij} \rangle_{i,j \leqslant l} \to \mathbf{B}_\lambda(S)^{\mathrm{ab}}$ must be injective. But its kernel is $\Gamma_2(\mathbf{B}_\lambda(S)/\Gamma_\infty) = \mathcal{L}_2(\mathbf{B}_\lambda(S))$, which must then be trivial.

The sphere. If $S = \mathbb{S}^2$, then $\mathbf{B}_\lambda(\mathbb{S}^2)$ is a quotient of $\mathbf{B}_\lambda$ (see Corollary A.22), which implies that its lower central series also stops at Γ_2.

Non-planar orientable surfaces. Proposition 6.33 deals with most of them; the remaining ones are the closed surfaces, for which Proposition 6.34 is the relevant statement.

Non-orientable surfaces. These are dealt with in Corollary 6.37, except for the closed ones, whose Lie ring is studied separately, in Corollary 6.39. $\qquad\square$

The following observation will be of essence in our study of the Lie ring of $\mathbf{B}_\lambda(S)$ when S is not planar or the sphere:

Fact 6.32 *If S has a handle or a crosscap, then $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ identifies with the kernel $\langle s_i^2, a_{ij} \rangle_{i,j \leqslant l}$ of the central extension of Theorem 6.28.*

Proof. We have already observed that $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ identifies with $\Gamma_2(\mathbf{B}_\lambda(S)/\Gamma_\infty)$, which is contained in $\langle s_i^2, a_{ij} \rangle_{i,j \leqslant l}$ (see the beginning of the proof of Theorem 6.30). If S has a handle or a crosscap, then the quotient $(\pi_1(S)^{\mathrm{ab}} \times \mathbb{Z}/2)^l$ of $\mathbf{B}_\lambda(S)/\Gamma_\infty$ identifies with its abelianisation (see Proposition 6.26). As a consequence, the subgroup $\Gamma_2(\mathbf{B}_\lambda(S)/\Gamma_\infty)$ is the whole of $\langle s_i^2, a_{ij} \rangle_{i,j \leqslant l}$. $\qquad\square$

Non-planar orientable surfaces. We first turn our attention to the case where our surface is not closed.

Proposition 6.33 *If S is a non-planar orientable surface which is not closed, then $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ is free abelian on the a_{ij} and the s_i^2.*

Proof. We already know that these elements generate $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ (Fact 6.32), so we need to show that they are linearly independent. The argument from the proof of Proposition 3.4 works here, using the maps from $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ to $\mathcal{L}_2(\mathbf{B}_{n_i}(S)) \cong \mathbb{Z}$ and $\mathcal{L}_2(\mathbf{B}_{n_i+n_j}(S)) \cong \mathbb{Z}$ instead of the maps from $\mathbf{B}_\lambda^{\mathrm{ab}}$ to $\mathbf{B}_{n_i}^{\mathrm{ab}}$ and $\mathbf{B}_{n_i+n_j}^{\mathrm{ab}}$. This uses the fact that $\mathcal{L}_2(\mathbf{B}_n(S)) \cong \mathbb{Z}$ if $n \geqslant 3$, from Theorem 6.19. $\qquad\square$

The case of closed orientable surfaces is a bit trickier. In fact, we first need to generalise [BGG08, Th. 1]:

Proposition 6.34 *Let $g \geqslant 1$. Then $\mathcal{L}_2(\mathbf{B}_\lambda(\Sigma_g))$ is the quotient of the free abelian group on the a_{ij} and the s_i^2 by the relations:*

$$(n_i + g - 1)s_i^2 + \sum_{j \neq i} n_j a_{ij} = 0, \quad \text{for all } 1 \leqslant i \leqslant n.$$

Proof. Let us consider the subsurface $\Sigma_{g,1}$ of Σ_g obtained by removing an open disc. The corresponding embedding of $\Sigma_{g,1}$ into Σ_g induces the quotient maps described in Corollary A.22. In particular, $\mathbf{B}_\lambda(\Sigma_g)/\Gamma_3$ is the quotient of $\mathbf{B}_\lambda(\Sigma_{g,1})/\Gamma_3$ by the classes of the relations from Corollary A.22. Since the $A_{\alpha\beta}$ are in $\Gamma_2(\mathbf{B}_\lambda(\Sigma_{g,1}))$, these relations are between elements of Γ_2, that is, of $\langle s_i^2, a_{ij} \rangle_{i,j \leqslant l}$. In order to write them as relations between the s_i^2 and the a_{ij}, we need to recall that if α is in the i-th block of λ, for any $r \leqslant g$, we have, in $\mathcal{L}_2(\mathbf{B}_\lambda(S))$:

$$\overline{\left[a_r^{(\alpha)}, (b_r^{(\alpha)})^{-1}\right]} = -\overline{\left[a_r^{(\alpha)}, b_r^{(\alpha)}\right]} = -\overline{\left[\sigma_\alpha b_r^{(\alpha)} \sigma_\alpha^{-1}, (a_r^{(\alpha)})^{-1}\right]} = -\overline{\sigma_\alpha^2} = -s_i^2,$$

where the third equality comes from Figure 6.2. Moreover, the class of $A_{\alpha\beta}$ modulo $\Gamma_3(\mathbf{B}_\lambda(S))$ is either s_i^2 if α and β are both in the i-th block of λ, or a_{ij} if α is in the i-th block and β is in the j-th block for some $j \neq i$. Finally, using additive notations in the central subgroup $\langle s_i^2, a_{ij} \rangle$, the classes of the relations from Corollary A.22 are:

$$(n_i - 1)s_i^2 + \sum_{j \neq i} n_j a_{ij} = -g s_i^2 \quad \text{for all } 1 \leqslant i \leqslant n.$$

Moreover, these relators are central in $\mathbf{B}_\lambda(\Sigma_g)/\Gamma_\infty$, so their normal closure is only the subgroup they generate, which implies that $\Gamma_2(\mathbf{B}_\lambda(\Sigma_g)/\Gamma_3)$ is the quotient of $\Gamma_2(\mathbf{B}_\lambda(\Sigma_{g,1})/\Gamma_3)$ by these relations. By Proposition 6.33, the latter is free on the s_i^2 and the a_{ij}. Whence our claim. $\qquad\square$

Remark 6.35 Let us remark that this computation of $\mathcal{L}_2(\mathbf{B}_\lambda(\Sigma_g))$ is very similar to the computation of $\mathcal{L}_1(\mathbf{B}_\lambda(\mathbb{S}^2))$ in the proof of Proposition 6.26 (note that s_i^2 was equal to $2s_i$ there, but here s_i and s_i^2 do not live in the same part of the Lie ring). The only difference lies in the degree of our relations with respect with the lower central series: if the surface has a handle, then the pure braid generators $A_{\alpha\beta}$ belongs to the derived subgroup (see Figure 6.2).

Non-orientable surfaces. For non-orientable surfaces, we already know that the s_i^2 vanish in the quotient by Γ_∞, as in the proof of Theorem 6.19. As a consequence, $\Gamma_2(\mathbf{B}_n(S)/\Gamma_\infty)$ is generated by the a_{ij}. The following proposition is an analogue of [BGG17, Prop. 3.7] in the non-orientable case:

Proposition 6.36 *Let $g \geqslant 0$ and $m, n \geqslant 3$. We have:*

$$\mathbf{B}_{m,n}(\mathcal{N}_{g,1})/\Gamma_\infty \cong (\mathbb{Z}/2)^2 \times (\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g,$$

where the factors are respectively generated by s_1 and s_2, a_{12}, c_r' and c_r. The action is given by $c_r c_r' c_r^{-1} = c_r' a_{12}$ (for all $r \leqslant g$), and all the other pairs of generators commuting. In particular, it is 2-nilpotent, and $\mathcal{L}_2(\mathbf{B}_{m,n}(\mathcal{N}_{g,1}))$ is infinite cyclic, generated by a_{12}.

Proof. Recall that we have a split extension (from Proposition 6.25):

$$\mathbf{B}_m(\mathcal{N}_{g,1+n}) \hookrightarrow \mathbf{B}_{m,n}(\mathcal{N}_{g,1}) \twoheadrightarrow \mathbf{B}_n(\mathcal{N}_{g,1}).$$

Thus, we can get a presentation of $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})$ from the presentation of the quotient described in Proposition A.15 and the presentation of the kernel from Proposition A.16, using the method of §A.3. The generators of this presentation are the σ_i for $i \neq m$, the $c_r = c_r^{(1)}$, the $c_r' := c_r^{(m+1)}$, and the $z_j = A_{1,m+j}$ (where our conventions are those from §A.2).

We use this to get a presentation of the quotient of $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})$ by the normal closure N of the $\sigma_i \sigma_{i+1}^{-1}$ for $i < m+n$ and $i \notin \{m-1, m\}$, together with the $z_j z_{j+1}^{-1}$ for $j < n$. This quotient is generated by the common class s_1 (resp. s_2, resp. a_{12}) of $\sigma_1, ..., \sigma_{m-1}$, the common class s_2 of $\sigma_{m+1}, ..., \sigma_{m-1}$, and the common class a_{12} of $z_1, ..., z_n$, together with the c_r and the c_r' for $r \leqslant g$. They are subject to the following relations:

- Relations coming from those of $\mathbf{B}_n(\mathcal{N}_{g,1})$: s_2 commutes with the c_r' ($BN1$), the c_r' commute with one another ($BN2$) and $s_2^2 = 1$ ($BN3$).
- Relations coming from those of $\mathbf{B}_m(\mathcal{N}_{g,1+n})$: s_1 commutes with the c_r ($BN1$), the c_r commute with one another ($BN2$), $s_1^2 = 1$ ($BN3$), a_{12} commutes with s_1 ($BN4$), and with the c_r ($BN5$). ($BN6$) and ($BN7$) become trivial.
- Relations describing the action by conjugation of s_2 and the c_r' on s_1, a_{12} and the c_r. This action is easily seen to be trivial in most cases, since most of the pairs of elements involved come from elements having disjoint support in $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})$, hence they commute. Namely, this holds for c_s and c_r' when $r < s$: using the fact that a_{12} commutes with c_s, we see that c_s is the classes of $z_1 c_r^{(1)} z_1^{-1}$, whose support is disjoint from the support of $c_r^{(m+1)}$ (up to isotopy). This is again true for a_{12} and the c_r', since s_2 commutes with the c_r': c_r' is the class of $c_r^{(m+2)} = \sigma_{m+1} c_r^{(m+1)} \sigma_{m+1}^{-1}$, whose support is disjoint from that of z_1. Finally, the only pair of generators under scrutiny for which this does not hold are c_r and c_r' (for $r \leqslant g$). But for them, the situation is the one from Figure 6.1: $[(c_r')^{-1}, c_r] = \overline{A}_{1,m+1} = a_{12}$.

These relations can be summed up as:

- All the generators commute pairwise, except c_r and c_r' for $r \leqslant g$,
- $s_1^2 = s_2^2 = 1$,
- $c_r c_r' c_r^{-1} = c_r' a_{12}$.

This is a presentation of the group described in the statement. Its commutator subgroup is the factor $\mathbb{Z}$ of the decomposition, which is infinite cyclic, generated by a_{12}. It is also central, hence the group is 2-nilpotent. Since this group is $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})/N$, N contains $\Gamma_3(\mathbf{B}_{m,n}(\mathcal{N}_{g,1}))$. But the elements normally generating N are in $\Gamma_\infty(\mathbf{B}_{m,n}(\mathcal{N}_{g,1}))$; see Lemma 6.27 and its proof. Thus $N = \Gamma_\infty(\mathbf{B}_{m,n}(\mathcal{N}_{g,1}))$, and the Lie ring of the quotient $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})/N$ identifies with the Lie ring of $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})$. In particular, since its Γ_3 is trivial, its $\mathcal{L}_2$ coincides with its Γ_2, which is infinite cyclic, generated by a_{12}. $\square$

Corollary 6.37 *Let S be a non-orientable surface which is not closed. Let $\lambda = (n_1, ..., n_l)$ be a partition whose blocks are of size at least 3. Then $\mathcal{L}_2(\mathbf{B}_\lambda(S))$ is free abelian on the a_{ij} for $1 \leqslant i < j \leqslant l$.*

Proof. We first show that if $m, n \geqslant 3$, $\mathcal{L}_2(\mathbf{B}_{m,n}(S))$ is infinite cyclic, generated by a_{12}. This is true for $S = \mathcal{N}_{g,1}$ by Proposition 6.36. Then we can follow the same method as in the proof of Theorem 6.19, to which the reader is referred for more details. Namely, we already know that $\mathcal{L}_2(\mathbf{B}_{m,n}(S))$ is generated by a_{12}, and we need to show that it has infinite order. If S is compact, we can embed S into some $\mathcal{N}_{g,1}$, and the image of a_{12} by the corresponding morphism from $\mathcal{L}_2(\mathbf{B}_{m,n}(S))$ to $\mathcal{L}_2(\mathbf{B}_{m,n}(\mathcal{N}_{g,1}))$ has infinite order, whence our claim in this case. Then, if S is not compact, a relation saying that a_{12} has finite order in $\mathbf{B}_{m,n}(S)/\Gamma_3$ would hold in a compact subsurface, which is impossible by the previous case.

From this, we can deduce the result for λ having more than two blocks, reasoning as in the proofs of Propositions 3.4 and 6.33. Indeed, each canonical map $\mathcal{L}_2(\mathbf{B}_\lambda(S)) \to \mathcal{L}_2(\mathbf{B}_{n_i,n_j}(S)) \cong \mathbb{Z}$ kills

all the a_{kl}, except a_{ij}, which is sent to a generator of the target. Thus the a_{ij} must be linearly independent. $\square$

Proposition 6.38 *Let $g \geqslant 0$ and $m, n \geqslant 3$. Then $\mathcal{L}_2(\mathbf{B}_{m,n}(\mathcal{N}_g))$ is cyclic of order 2, generated by a_{12}.*

Proof. Let us consider $G := \mathbf{B}_{m,n}(\mathcal{N}_g)/\Gamma_3$. This group is the quotient of $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})/\Gamma_3 = \mathbf{B}_{m,n}(\mathcal{N}_g)/\Gamma_\infty \cong (\mathbb{Z}/2)^2 \times (\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$ described in Proposition 6.36 by the images of the boundary relations from Corollary A.22. Before considering these relations, we can already remark that the commutator subgroup of $\mathbf{B}_{m,n}(\mathcal{N}_{g,1})/\Gamma_3$ is cyclic generated by a_{12}, so the same holds for G. Since $\Gamma_2(G)$ identifies with $\mathcal{L}_2(\mathbf{B}_{m,n}(\mathcal{N}_g))$, we only need to show that a_{12} has order 2 in G to prove our statement.

We recall that $A_{\alpha\beta}$ is sent to $s_1^2 = s_2^2 = 1$ if α and β are in the same block of the partition (m, n), and to a_{12} if they are not. As a consequence, these relations are:

$$c_1^2 \cdots c_g^2 = a_{12}^m \quad \text{and} \quad (c_1')^2 \cdots (c_g')^2 = a_{12}^n.$$

Since c_1 commutes with all the other generators except c_1', and $[c_1, c_1'] = a_{12}$, by applying the commutator with c_1 to the second relation, we get $a_{12}^2 = 1$. In particular, G is in fact a quotient of $(\mathbb{Z}/2)^2 \times ((\mathbb{Z}/2) \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$, and the above relations (and thus the structure of $\mathbf{B}_{m,n}(\mathcal{N}_g)/\Gamma_3$) depend only on the parity of m and n. Moreover, since $a_{12}^2 = 1$, we are left with showing that a_{12} is not trivial in G.

Suppose first than m and n are both even. Then the relations become $c_1^2 \cdots c_g^2 = (c_1')^2 \cdots (c_g')^2 = 1$. Thus, if A denotes the abelian group $\pi_1(\mathcal{N}_g)^{\mathrm{ab}} \cong \mathbb{Z}^g/(2, 2, ..., 2)$, we have $G \cong (\mathbb{Z}/2)^2 \times ((\mathbb{Z}/2) \times A) \rtimes A$, whose commutator subgroup is cyclic of order 2, generated by a_{12}.

If m and n have different parities, we can assume (by symmetry) that m is odd and n is even. Then the relations become $(c_1')^2 \cdots (c_g')^2 = 1$ and $c_1^2 \cdots c_g^2 = a_{12}$. In $(\mathbb{Z}/2)^2 \times ((\mathbb{Z}/2) \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$, the cyclic subgroup generated by $c_1^2 \cdots c_g^2 a_{12}^{-1}$ is normal (it is even central). Thus, if we denote by $\tilde{A}$ the abelian group $\mathbb{Z}^g/(4, 4, ..., 4)$, we have $G \cong (\mathbb{Z}/2)^2 \times \tilde{A} \rtimes A$, whose commutator subgroup is cyclic of order 2, generated by a_{12} (which identifies with the class of $(2, 2, ..., 2)$ in $\tilde{A}$).

If m and n are both odd, then the relations become $c_1^2 \cdots c_g^2 = (c_1')^2 \cdots (c_g')^2 = a_{12}$. In this case, there is no obvious semi-direct product decomposition of G where a_{12} is clearly non-trivial, so we need another argument to show that $a_{12} \neq 1$. If $g = 1$, one can see that $G \cong (\mathbb{Z}/2)^2 \times Q_8$, where c_1 is sent to i, c_1' is sent to j, and a_{12} identifies with the central element -1 of the quaternion group Q_8. For $g \geqslant 1$, we can find a similar quotient of G, by considering the quotient Q_8^g/H, where H is the hyperplane of $\mathcal{Z}(Q_8^g) \cong (\mathbb{Z}/2)^g$ defined by the vanishing of the sum of the coordinates. Since H is central in Q_8^g, it is normal. Moreover, Q_8^g decomposes as a central extension of $(\mathbb{Z}/2)^{2g}$ by $(\mathbb{Z}/2)^g$, which induces a central extension:

$$(\mathbb{Z}/2)^g/H \cong \mathbb{Z}/2 \longhookrightarrow Q_8^g/H \longtwoheadrightarrow (\mathbb{Z}/2)^{2g}.$$

Using the presentation of G, we can see that there is a well-defined projection from G onto Q_8^g/H sending c_r to $(1, ..., 1, i, 1, ..., 1)$, c_r' to $(1, ..., 1, j, 1, ..., 1)$ (where the non-trivial coordinate is the r-th one in both cases), and s_1 and s_2 to 1. This projection sends $a_{12} = [c_1, c_1']$ to the generator of the center $(\mathbb{Z}/2)^g/H \cong \mathbb{Z}/2$. Thus a_{12} is again not trivial in G, whence our claim. $\square$

Corollary 6.39 *Let $g \geqslant 0$ and let $\lambda = (n_1, ..., n_l)$ be a partition whose blocks are of size at least 3. Then $\mathcal{L}_2(\mathbf{B}_\lambda(\mathcal{N}_g)) \cong (\mathbb{Z}/2)^{l(l-1)/2}$ is the free elementary abelian 2-group on the a_{ij} for $1 \leqslant i < j \leqslant l$.*

Proof. The proof is again the same as the proofs of Propositions 3.4 and 6.33, using the canonical maps $\mathcal{L}_2(\mathbf{B}_\lambda(\mathcal{N}_g)) \to \mathcal{L}_2(\mathbf{B}_{n_i,n_j}(\mathcal{N}_g)) \cong \mathbb{Z}/2$, where the later is generated by the image of a_{ij}, by Proposition 6.38. $\square$

6.5 Partitions with small blocks

Now that we have a complete description of the lower central series of $\mathbf{B}_\lambda(S)$ in the stable case, that is, when the blocks of the partition λ have size at least 3 (§6.4), we turn our attention to the cases when λ does have blocks of size 1 or 2. Then we ask ourselves: under this assumption, when does the lower central series of $\mathbf{B}_\lambda(S)$ stop? For most surfaces, it does not; see Proposition 6.40. In fact, there are only six surfaces to which this result does not apply. One of them is the disc, for which an answer has already been given in §3. Another one is the cylinder, which case can easily be deduced from the case of the disc. Four surfaces remain: the torus $\mathbb{T}^2$ (see §6.5.2), the Möbius strip $\mathbb{M}^2$ (see §6.5.3), the sphere $\mathbb{S}^2$ (see §6.5.4) and the projective plane $\mathbb{P}^2$ (see §6.5.5).

6.5.1 The generic cases

As a direct corollary of Propositions 6.16 and 6.17, we get the following result:

Proposition 6.40 *Let $\lambda = (n_1, ..., n_l)$ be a partition of an integer $n \geqslant 1$.*

- *If λ has at least one block of size 1 and $\pi_1(S)$ is not abelian (that is, if we suppose that $S \notin \{\mathbb{D} - pt, \mathbb{D}, \mathbb{S}^2, \mathbb{T}^2, \mathbb{P}^2, \mathbb{M}^2\}$ up to isotopy equivalence), then the lower central series of $\mathbf{B}_\lambda(S)$ does not stop.*
- *If λ has at least one block of size 2 and $\pi_1(S)^{\mathrm{ab}}$ is not finite (that is, $S \notin \{\mathbb{D}, \mathbb{S}^2, \mathbb{P}^2\}$), then the lower central series of $\mathbf{B}_\lambda(S)$ does not stop.*

Proof. In the first case, there is a surjection $\mathbf{B}_\lambda(S) \twoheadrightarrow \mathbf{B}_1(S) \cong \pi_1(S)$, and in the second one, a surjection $\mathbf{B}_\lambda(S) \twoheadrightarrow \mathbf{B}_2(S)$. Propositions 6.16 and 6.17 say that, under our hypotheses, the lower central series of the quotient does not stop in either case. The result then follows from Lemma 1.1. $\qquad\square$

Thus the question of whether the lower central series of $\mathbf{B}_\lambda(S)$ stops has been answered for every partition, except for the six surfaces $\mathbb{D} - pt$, $\mathbb{D}$, $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{P}^2$ and $\mathbb{M}^2$. In fact, $\mathbf{B}_\lambda(\mathbb{D}) = \mathbf{B}_\lambda$ has already been considered in §3; see Theorem 3.5. Also, since $\mathbf{B}_1(\mathbb{D}) = \{1\}$, we have an isomorphism $\mathbf{B}_\lambda(\mathbb{D} - pt) \cong \mathbf{B}_{1,\lambda}(\mathbb{D})$ for every partition λ by Proposition 6.25, so that we can deduce the remaining answer for $\mathbb{D} - pt$ from the answer for the disc. Namely, Lemma 3.7 and Proposition 3.9 imply:

Lemma 6.41 *If λ has at least two blocks of size 1, then the lower central series of $\mathbf{B}_\lambda(\mathbb{D} - pt)$ does not stop. If $\lambda = (1, n_2, \ldots, n_l)$ where every $n_i \geqslant 3$, then its lower central series stops at Γ_2.*

Therefore, we are left with four remaining cases: the torus, the Möbius strip, the sphere and the projective plane.

6.5.2 Partitioned braids on the torus

We know that the lower central series of $\mathbf{B}_\lambda(\mathbb{T}^2)$ stops if λ has only blocks of size at least 3, and that it does not if there is at least one block of size 2. The remaining cases are dealt with using the following generalisation of [BGG08, Lem. 17]:

Proposition 6.42 *Let μ be any partition. There is an isomorphism $\mathbf{B}_{1,\mu}(\mathbb{T}^2) \cong \mathbf{B}_\mu(\mathbb{T}^2 - pt) \times \mathbb{Z}^2$.*

Proof. Proposition 6.25 gives a short exact sequence:

$$\mathbf{B}_\mu(\mathbb{T}^2 - pt) \lhook\joinrel\longrightarrow \mathbf{B}_{1,\mu}(\mathbb{T}^2) \longrightarrow\!\!\!\!\!\rightarrow \pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2.$$

Let $n \geqslant 2$ such that μ is a partition of $n-1$. The center of $\mathbf{B}_n(\mathbb{T}^2)$ is generated by two braids α and β corresponding to rotating all the punctures along each factor of $\mathbb{S}^1 \times \mathbb{S}^1$; see [PR00, Prop. 4.2]. These are pure braids, so they are in the subgroup $\mathbf{B}_{1,\mu}(\mathbb{T}^2)$ of $\mathbf{B}_n(S)$. The above projection (forgetting all strands but one) maps these two elements to a basis of $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$. As

a consequence, it restricts to an isomorphism $\langle \alpha, \beta \rangle \cong \mathbb{Z}^2$. Thus the above short exact sequence splits, and the corresponding action of $\mathbb{Z}^2$ is trivial (α and β being central), hence it is in fact a direct product. $\qquad \square$

It is then an easy task to finish proving the following:

Theorem 6.43 *Let λ be a partition of an integer $n \geqslant 1$. The lower central series of $\mathbf{B}_\lambda(\mathbb{T}^2)$:*

- *does not stop if λ has at least two blocks of size 1 or at least one block of size 2.*
- *stops at Γ_3 in all the other cases, except for $\mathbf{B}_1(\mathbb{T}^2) \cong \mathbb{Z}^2$.*

Proof. If λ has at least two blocks of size 1, then $\mathbf{B}_\lambda(\mathbb{T}^2)$ surjects onto $\mathbf{B}_{1,1}(\mathbb{T}^2)$, which is isomorphic to $\mathbf{B}_1(\mathbb{T}^2 - pt) \times \mathbb{Z}^2 \cong \mathbf{F}_2 \times \mathbb{Z}^2$ by Proposition 6.42. The lower central series of $\mathbf{F}_2$ does not stop, whence the result in this case, by Lemma 1.1.

If λ has exactly one block of size 1 and no block of size 2, then Proposition 6.42 gives an isomorphism $\mathbf{B}_\lambda(\mathbb{T}^2) \cong \mathbf{B}_\mu(\mathbb{T}^2 - pt) \times \mathbb{Z}^2$ where μ has only blocks of size at least 3. Then Corollary 6.29 implies that the lower central series stops at most at Γ_3 in this case. In fact, if μ is non-trivial, then the lower central series of $\mathbf{B}_\mu(\mathbb{T}^2 - pt)$ stops exactly at Γ_3; see Theorem 6.30.

The other cases have already been treated as part of Corollary 6.29, Theorem 6.30 and Proposition 6.40. $\qquad \square$

6.5.3 Partitioned braids on the Möbius strip

As in the case of the torus, we know that the lower central series of $\mathbf{B}_\lambda(\mathbb{M}^2)$ stops if λ has only blocks of size at least 3, and that it does not if there is at least one block of size 2. The only remaining cases are the ones when λ has some blocks of size 1, the other ones being of size at least 3. We begin by showing that the lower central series does not stop when there are at least two blocks of size 1. In this case, $\mathbf{B}_\lambda(\mathbb{M}^2)$ surjects onto $\mathbf{B}_{1,1}(\mathbb{M}^2) = \mathbf{P}_2(\mathbb{M}^2)$, so this case follows by Lemma 1.1 from the following study of $\mathbf{P}_2(\mathbb{M}^2)$:

Lemma 6.44 *A presentation of the pure braid group $\mathbf{P}_2(\mathbb{M}^2)$ is given by generators γ_1, γ_2, A subject to the relations*

$$\begin{cases} \gamma_1 A \gamma_1^{-1} = \gamma_2^{-1} A^{-1} \gamma_2 \\ \gamma_1 \gamma_2 \gamma_1^{-1} = \gamma_2^{-1} A^{-1} \gamma_2^2. \end{cases}$$

Proof. Proposition 6.25 gives a decomposition $\mathbf{P}_2(\mathbb{M}^2) \cong \pi_1(\mathbb{M}^2 - pt) \rtimes \pi_1(\mathbb{M}^2) \cong \mathbf{F}_2 \rtimes \mathbb{Z}$, where the projection onto $\pi_1(\mathbb{M}^2) \cong \mathbb{Z}$ is given by forgetting one strand (say, the first one). Then the factor $\mathbb{Z}$ is generated by $\gamma_1 := c_1^{(1)}$, and a free basis of the factor $\mathbf{F}_2$ is given by $\gamma_2 := c_1^{(2)}$ and $A = A_{12}$ (where we use notations from §A.2). Moreover, the action of γ_1 by conjugation on $\langle A, \gamma_2 \rangle$ is not difficult to compute. Namely, we have $\gamma_1 A \gamma_1^{-1} = \gamma_2^{-1} A^{-1} \gamma_2$; see Figure 6.3. Then, from Figure 6.1, we get that $[\gamma_2, \gamma_1^{-1}] = A$, which is equivalent to $\gamma_1 \gamma_2 \gamma_1^{-1} = \gamma_1 A \gamma_1^{-1} \gamma_2$. Using the previous relation, the latter equals $\gamma_2^{-1} A^{-1} \gamma_2^2$. These relations determine $\mathbf{P}_2(\mathbb{M}^2)$, since the group G that they define decomposes as $\langle A, \gamma_2 \rangle \rtimes \langle \gamma_1 \rangle$, and the obvious projection of G onto $\mathbf{P}_2(\mathbb{M}^2)$ must be an isomorphism on both the kernel and the quotient, hence it must be an isomorphism. $\qquad \square$
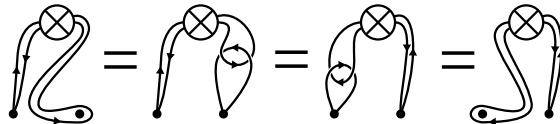

Figure 6.3 Relation in $\mathbf{P}_2(\mathbb{M}^2)$: $\gamma_1 A \gamma_1^{-1} = \gamma_2^{-1} A^{-1} \gamma_2$.

Corollary 6.45 *The lower central series of $\mathbf{P}_2(\mathbb{M}^2)$ does not stop.*

Proof. Let us consider the quotient of $\mathbf{P}_2(\mathbb{M}^2) = \mathbf{F}_2 \rtimes \mathbb{Z}$ by $\Gamma_2(\mathbf{F}_2)$. It is $\mathbb{Z}^2 \rtimes \mathbb{Z}$, where the generator $\overline{\gamma}_1$ of $\mathbb{Z}$ acts via the involution τ sending $\overline{A}$ to $-\overline{A}$ and $\overline{\gamma}_2$ to $\overline{\gamma}_2 - \overline{A}$. Then $V := \operatorname{Im}(\tau - 1)$ is $\mathbb{Z} \cdot \overline{A}$. By Proposition B.10, for $i \geqslant 2$, we have $\Gamma_i(\mathbb{Z}^2 \rtimes \mathbb{Z}) = 2^{i-2}V$, so the lower central series of $\mathbb{Z}^2 \rtimes \mathbb{Z}$ does not stop. Thus the lower central series of $\mathbf{P}_2(\mathbb{M}^2)$ does not either. $\qquad\square$

The answer for the remaining cases are consequences of the following result:

Proposition 6.46 *For $m \geqslant 3$ an integer, the lower central series of $\mathbf{B}_{1,m}(\mathbb{M}^2)$ does not stop.*

Proof. We can use Proposition 6.25 to get a decomposition:

$$\mathbf{B}_{1,m}(\mathbb{M}^2) \cong \mathbf{B}_m(\mathbb{M}^2 - pt) \rtimes \pi_1(\mathbb{M}^2),$$

where $\pi_1(\mathbb{M}^2) \cong \mathbb{Z}$ is identified with the subgroup of $\mathbf{B}_{1,m}(\mathbb{M}^2)$ generated by $\gamma_1 := c_1^{(1)}$. We know that $\Gamma_\infty(\mathbf{B}_{1,m}(\mathbb{M}^2))$ contains $\Gamma_\infty(\mathbf{B}_m(\mathbb{M}^2 - pt))$. The latter is fully invariant in $\mathbf{B}_m(\mathbb{M}^2 - pt)$, hence normal in $\mathbf{B}_{1,m}(\mathbb{M}^2)$, and we can consider the quotient

$$G := \mathbf{B}_{1,m}(\mathbb{M}^2)/\Gamma_\infty(\mathbf{B}_m(\mathbb{M}^2 - pt)) \cong (\mathbf{B}_m(\mathbb{M}^2 - pt)/\Gamma_\infty) \rtimes \mathbb{Z}.$$

Moreover, Lemma 6.12 and Remark 6.14 give a central extension:

$$\langle \sigma \rangle \longhookrightarrow \mathbf{B}_m(\mathbb{M}^2 - pt)/\Gamma_\infty \longtwoheadrightarrow \pi_1(\mathbb{M}^2 - pt)^{\mathrm{ab}}.$$

The element σ is the common class of the usual generators σ_α of $\mathbf{B}_m$. Since these commute with γ_1 in $\mathbf{B}_{1,m}(\mathbb{M}^2)$, σ is central not only in $\mathbf{B}_m(\mathbb{M}^2 - pt)/\Gamma_\infty$, but in fact in G. In particular, $\langle \sigma \rangle$ is normal in G, and the quotient decomposes as:

$$G/\sigma \cong \left[(\mathbf{B}_\mu(\mathbb{M}^2 - pt)/\Gamma_\infty)/\sigma \right] \rtimes \pi_1(\mathbb{M}^2) \cong \pi_1(\mathbb{M}^2 - pt)^{\mathrm{ab}} \rtimes \pi_1(\mathbb{M}^2) \cong \mathbb{Z}^2 \rtimes \mathbb{Z}.$$

A basis of the $\mathbb{Z}^2$ factor is given by $\overline{\gamma_2}$ and $A = \overline{A_{12}}$. Consider the morphism $\mathbf{P}_2(\mathbb{M}^2) \to \mathbf{B}_{1,m}(\mathbb{M}^2)$ corresponding to adding $m - 1$ trivial strands to the second block (constructed as in the proof of Proposition 6.25). Composing with the projection, we get a morphism from $\mathbf{P}_2(\mathbb{M}^2) = \mathbf{F}_2 \rtimes \mathbb{Z}$ to $G/\sigma = \mathbb{Z}^2 \rtimes \mathbb{Z}$. From the explicit description of both these groups, one easily sees that it has to induce an isomorphism $\mathbf{P}_2(\mathbb{M}^2)/\Gamma_2(\mathbf{F}_2) \cong G/\sigma$. But the lower central series of $\mathbf{P}_2(\mathbb{M}^2)/\Gamma_2(\mathbf{F}_2)$ does not stop (see the proof of Corollary 6.45), so we have found a quotient of $\mathbf{B}_{1,m}(\mathbb{M}^2)$ whose lower central series does not stop, whence the result. $\qquad\square$

Let us sum up our results about the lower central series of $\mathbf{B}_\lambda(\mathbb{M}^2)$:

Theorem 6.47 *Let $\lambda = (n_1, ..., n_l)$ be a partition of an integer $n \geqslant 1$. The lower central series of $\mathbf{B}_\lambda(\mathbb{M}^2)$:*

- *does not stop if λ has at least one block of size 1 or 2, except for $\mathbf{B}_1(\mathbb{M}^2) \cong \mathbb{Z}$.*
- *stops in all the other cases, at Γ_3 if $l \geqslant 2$ and at Γ_2 if $l = 1$.*

Proof. The first statement follows from Proposition 6.40 (for blocks of size 2) and from Corollary 6.45 and Proposition 6.46 (for blocks of size 1). The second one is part of the general results of Corollary 6.29 and Theorem 6.30. $\qquad\square$

6.5.4   Partitioned braids on the sphere

For any partition $\lambda = (n_1, ..., n_l)$, the inclusion of the disc into the sphere induces a surjection $\mathbf{B}_\lambda \twoheadrightarrow \mathbf{B}_\lambda(\mathbb{S}^2)$ by Corollary A.22. We can thus apply Lemma 1.1 to deduce that the lower central series of $\mathbf{B}_\lambda(\mathbb{S}^2)$ stops whenever the one of $\mathbf{B}_\lambda$ does. Namely, by Theorem 3.5, it stops at Γ_2 if $n_i \geqslant 3$ for all i, save at most two indices for which $n_i = 1$. Since the abelianisation of $\mathbf{B}_\lambda(\mathbb{S}^2)$ has already been computed in Proposition 6.26, we have a complete description of the lower central series in these cases.

The above argument can be extended somewhat if we remark that, because of Proposition 6.25, $\mathbf{B}_1(\mathbb{S}^2) = \pi_1(\mathbb{S}^2) = \{1\}$ is the cokernel of the canonical morphism $\mathbf{B}_\mu(\mathbb{D}) \to \mathbf{B}_{1,\mu}(\mathbb{S}^2)$ (for any partition μ), which is thus surjective. As a consequence, the lower central series of $\mathbf{B}_\lambda(\mathbb{S}^2)$ also stops at Γ_2 when λ has three blocks of size 1, the other ones being of size at least 3. This proves the results of the first point in the following theorem, the other ones being the point of the remainder of this section.

Theorem 6.48 *Let $n \geqslant 1$ be an integer, let $\lambda = (n_1, ..., n_l)$ be a partition of n. The lower central series of $\mathbf{B}_\lambda(\mathbb{S}^2)$:*

- *stops at Γ_2 if $n_i \geqslant 3$ for all i, save at most three indices for which $n_i = 1$.*
- *does not stop in all the other cases, except for $\mathbf{B}_2(\mathbb{S}^2) \cong \mathbb{Z}/2$, $\mathbf{B}_{2,1}(\mathbb{S}^2) \cong \mathbb{Z}/4$ and the $\mathbf{B}_{2,m}(\mathbb{S}^2)$ with $m \geqslant 3$, whose lower central series stops at $\Gamma_{v_2(m)+1}$ or at $\Gamma_{v_2(m)+2}$.*

Proof. The first part has just been proved. Let us indicate how we deduce the second one from the results below.

If λ has at least four blocks of size 1, then $\mathbf{B}_\lambda(\mathbb{S}^2)$ surjects onto the pure braid group $\mathbf{P}_4(\mathbb{S}^2)$ (by forgetting the other blocks). Using Lemma 1.1, the result in this case follows from Lemma 6.49.

The remaining cases are the ones where there is at least one block of size 2. If the partition has at least three blocks, the result follows from 6.50. If the partition has exactly two blocks, and if the size of the other block is at least 2, then we can apply either Proposition 6.51 or Proposition 6.53.

Finally, let us compute $\mathbf{B}_{2,1}(\mathbb{S}^2)$ and $\mathbf{B}_2(\mathbb{S}^2)$. As recalled in the proof of Lemma 6.49, $\mathbf{P}_3(\mathbb{S}^2) \cong \pi_1(SO_3(\mathbb{R})) \cong \mathbb{Z}/2$, so that $\mathbf{B}_{2,1}(\mathbb{S}^2)$ is an extension of $\mathfrak{S}_2 \cong \mathbb{Z}/2$ by $\mathbb{Z}/2$. Thus, it must be isomorphic to $(\mathbb{Z}/2)^2$ or to $\mathbb{Z}/4$. In order to decide between the two, one can use the computation of $\mathbf{B}_{2,1}^{ab}$ from Proposition 6.11, and find that $\mathbf{B}_{2,1}(\mathbb{S}^2) \cong \mathbb{Z}/4$. As for $\mathbf{B}_2(\mathbb{S}^2)$, we use Proposition 6.25 to get an exact sequence $\pi_1(\mathbb{S}^2 - \{pt\}) \to \mathbf{P}_2(\mathbb{S}^2) \to \pi_1(\mathbb{S}^2)$, which implies that $\mathbf{P}_2(\mathbb{S}^2) = 1$, whence $\mathbf{B}_2(\mathbb{S}^2) \cong \mathfrak{S}_2$. $\qquad \square$

Blocks of size 1. We need to consider the case of the pure braid group on four strands:

Lemma 6.49 *The lower central series of $\mathbf{P}_4(\mathbb{S}^2)$ does not stop.*

Proof. We give a sketch of proof of the decomposition of $\mathbf{P}_4(\mathbb{S}^2)$ given in [GG04b, Th. 4]. We recall that we have a short exact sequence by Proposition 6.25:

$$\mathbf{P}_1(\mathbb{S}^2 - \{3 \text{ pts}\}) \hookrightarrow \mathbf{P}_4(\mathbb{S}^2) \twoheadrightarrow \mathbf{P}_3(\mathbb{S}^2).$$

It is known that $\mathbf{P}_3(\mathbb{S}^2) \cong \mathbb{Z}/2$. In fact, an isomorphism between $\pi_1(SO_3(\mathbb{R}))$ and $\mathbf{P}_3(\mathbb{S}^2)$ is induced by $\varphi \mapsto (\varphi(e_1), \varphi(e_2), \varphi(e_3))$ from $SO_3(\mathbb{R})$ to $F_3(\mathbb{S}^2)$. Moreover, a splitting of the above short exact sequence is given by sending the generator of $\mathbf{P}_3(\mathbb{S}^2)$ to the full twist. Since the latter is central, $\mathbf{P}_4(\mathbb{S}^2)$ is the direct product of $\mathbb{Z}/2$ with $\pi_1(\mathbb{S}^2 - \{3 \text{ pts}\}) \cong \mathbf{F}_2$. Thus $\mathbf{P}_4(\mathbb{S}^2)$ is residually nilpotent (but not nilpotent), whence the result. $\qquad \square$

Blocks of size 2. Let us begin with the case where the partition has at least three blocks:

Proposition 6.50 *Let λ be a partition of n with at least three blocks, and at least one block of size 2. Then the lower central series of $\mathbf{B}_\lambda(\mathbb{S}^2)$ does not stop.*

Proof. Let μ be any partition. Let us consider the quotient $\mathbf{B}_{2,\mu}(\mathbb{S}^2)/\Gamma_2(\mathbf{B}_{1,1,\mu}(\mathbb{S}^2))$, which is (clearly) an extension:

$$\mathbf{B}_{1,1,\mu}(\mathbb{S}^2)^{ab} \longrightarrow G = \mathbf{B}_{2,\mu}(\mathbb{S}^2)/\Gamma_2(\mathbf{B}_{1,1,\mu}(\mathbb{S}^2)) \longrightarrow \mathfrak{S}_2. \qquad (6.2)$$

Let us fix $(n_1, ..., n_l) := (1, 1, \mu)$. From a presentation of the kernel (from Proposition 6.11) and a presentation of the quotient, using the method from §A.3, we can write down a presentation

of G. Precisely, G admits the presentation with generators s, s_i (for $1 \leqslant i \leqslant l$), and a_{ij} (for $1 \leqslant i < j \leqslant l$), subject to the following relations:

$$\begin{cases} (1) & s^2 = a_{12}, \\ (2) & s_i = 1 & \text{if } n_i = 1, \\ (3) & [s_i, s_j] = [s_i, a_{pq}] = [a_{pq}, a_{uv}] = 1 & \forall i, j, p, q, u, v, \\ (4) & [s, s_i] = 1 & \forall i \geqslant 1, \\ (5) & [s, a_{ij}] = 1 & \forall j > i \geqslant 3, \\ (6) & sa_{1j}s^{-1} = a_{2j} \text{ and } sa_{2j}s^{-1} = a_{1j} & \forall j \geqslant 3, \\ (7) & 2(n_i - 1)s_i + \sum_{j \neq i} n_j a_{ij} = 0 & \forall i \geqslant 1, \end{cases}$$

where the last relation uses additive notations in the abelian subgroup generated by the s_i and the a_{ij}.

Consider the subgroup H of G generated by s^2 together with the s_i and the a_{ij} for $i, j \geqslant 3$. One can easily see that it is central in G. Moreover, we can deduce from the presentation of G a presentation of G/H, which turns out to be very simple. Indeed, it is generated by s, the a_{1i} and the a_{2i} for $i \geqslant 3$. Moreover, since $n_1 = n_2 = 1$, for $i \geqslant 3$ the last relation becomes $a_{1i} + a_{2i} = 0$, which means that a_{1i} and a_{2i} are inverse to each other. For $i \in \{1, 2\}$, we find the relations $\sum n_j a_{1j} = 0$ and $\sum n_j a_{2j} = 0$, which are equivalent to each other modulo the previous relations. Thus, if we denote by a_i the element $a_{1i} = a_{2i}^{-1}$ of G/H, we get that G/H is generated by a_3, a_4, ..., a_l and s, subject to the relations:

$$\begin{cases} (1) & s^2 = 1 \\ (3) & [a_i, a_j] = 1 & \forall i, j, \\ (6) & sa_j s^{-1} = a_j^{-1} & \forall j \geqslant 3, \\ (7) & \sum n_j a_j = 0. \end{cases}$$

Finally, $G/H \cong A \rtimes (\mathbb{Z}/2)$, where A is the quotient of the free abelian group on $a_3, a_4, ..., a_l$ by the single relation $\sum n_j a_j = 0$, and $\mathbb{Z}/2$ acts on A via $-id$. We can use Corollary B.8 to compute the lower central series of G/H. Namely, since $A \cong \mathbb{Z}^{l-3} \times \mathbb{Z}/\gcd(n_i)$, it does not stop whenever $l > 3$. Thus, we have found a quotient of $\mathbf{B}_{2,\mu}$ having a non-stopping lower central series whenever μ has at least two blocks. $\qquad\square$

If λ has only two blocks, we first assume that the other block is large enough:

Proposition 6.51 *Let $m \geqslant 3$ be an integer. The lower central series of $\mathbf{B}_{2,m}(\mathbb{S}^2)$ stops at $\Gamma_{v_2(m)+1}$ or at $\Gamma_{v_2(m)+2}$, where v_2 is the 2-adic valuation.*

Proof. We have seen above that the lower central series of $\mathbf{B}_{1,1,m}(\mathbb{S}^2)$ (which is a quotient of $\mathbf{B}_{1,1,m}$) stops at Γ_2 if $m \geqslant 3$. Then, as in the proof of Proposition 3.11, we have that $\Gamma_\infty(\mathbf{B}_{2,m}(\mathbb{S}^2))$ contains $\Gamma_\infty(\mathbf{B}_{1,1,m}(\mathbb{S}^2)) = \Gamma_2(\mathbf{B}_{1,1,m}(\mathbb{S}^2))$. Then it is enough to show that the lower central series of $G := \mathbf{B}_{2,m}(\mathbb{S}^2)/\Gamma_2(\mathbf{B}_{1,1,m}(\mathbb{S}^2))$ stops. In order to do this, we use the calculations already made in the course of the proof of Proposition 6.50: the group G is a central extension of $(\mathbb{Z}/m) \rtimes (\mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts via $-id$. Thanks to Corollary B.8, we know that the lower central series of the latter is given by $\Gamma_i = 2^{i-1}\mathbb{Z}/m$ for $i \geqslant 2$. Hence it stops at Γ_2 if m is odd, and in general, it stops at $\Gamma_{v_2(m)+1}$. Now, we can use Corollary 1.3 to see that the lower central series of its central extension G stops at $\Gamma_{v_2(m)+1}$ or at $\Gamma_{v_2(m)+2}$. $\qquad\square$

Remark 6.52 It seems difficult to decide between the two possibilities, although our experimental calculations using GAP [GAP21] and the package NQ [Nic96] suggest that the lower central series of G (and hence also the one of $\mathbf{B}_{2,m}(\mathbb{S}^2)$) always stops at $\Gamma_{v_2(m)+2}$.

Finally, let us show that the lower central series of $\mathbf{B}_{2,2}(\mathbb{S}^2)$ does not stop, using Lemma 1.1. Namely, we are looking for a quotient of $\mathbf{B}_{2,2}(\mathbb{S}^2)$ whose lower central series does not stop. As a manageable non-abelian quotient, one can think of $\mathbf{B}_{2,2}(\mathbb{S}^2)/\Gamma_2(\mathbf{P}_4(\mathbb{S}^2))$, which is an extension

of $\mathfrak{S}_2 \times \mathfrak{S}_2$ by $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$. In fact, in order to make it even more manageable, we take a further quotient, turning it into a split extension to which the methods of the appendix apply.

Proposition 6.53 *The lower central series of* $\mathbf{B}_{2,2}(\mathbb{S}^2)$ *does not stop.*

Proof. We recall that $\Gamma_2(\mathbf{P}_4(\mathbb{S}^2))$ is fully invariant in $\mathbf{P}_4(\mathbb{S}^2)$, thus normal in $\mathbf{B}_{2,2}(\mathbb{S}^2)$, so the quotient $G = \mathbf{B}_{2,2}(\mathbb{S}^2)/\Gamma_2(\mathbf{P}_4(\mathbb{S}^2))$ is a well-defined extension of $\mathfrak{S}_2 \times \mathfrak{S}_2$ by $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$. The latter it is the abelian group generated by a_{ij} for $1 \leqslant i < j \leqslant 4$, subject to the four relations $\sum_{j \neq i} a_{ij} = 0$ (for each $i \leqslant 4$); this classical computation is part of Proposition 6.11.

The action of $\mathbf{B}_{2,2}(\mathbb{S}^2)$ on $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$ induced by conjugation factors through $\mathbf{B}_{2,2}(\mathbb{S}^2)/\mathbf{P}_4(\mathbb{S}^2) \cong \mathfrak{S}_2 \times \mathfrak{S}_2$. This action is by permutation of the indices of the generators a_{ij} of $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$. In particular, it fixes a_{12} and a_{34}, which then generate a central subgroup H of G. Let us consider the quotient by this central subgroup, that is, the quotient of $\mathbf{B}_{2,2}(\mathbb{S}^2)$ by its subgroup $\tilde{H} = \langle A_{12}, A_{34}, \Gamma_2(\mathbf{P}_4(\mathbb{S}^2)) \rangle$. This quotient is an extension:

$$\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}/H \lhook\joinrel\longrightarrow G/H \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{S}_2 \times \mathfrak{S}_2.$$

By definition of $\tilde{H}$, we have $\overline{\sigma}_1^2 = \overline{\sigma}_3^2 = 1$ in $\mathbf{B}_{2,2}(\mathbb{S}^2)/\tilde{H} = G/H$, so this short exact sequence splits. Moreover, $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}/H$ is the quotient of $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$ by the relations $a_{12} = a_{34} = 0$, modulo which the relations defining $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}$ become $a_{13} = -a_{14} = a_{13} = -a_{23}$. Thus $\mathbf{P}_4(\mathbb{S}^2)^{\mathrm{ab}}/H \cong \mathbb{Z}$, and $G/H \cong \mathbb{Z} \rtimes (\mathfrak{S}_2)^2$, where both transpositions act via a sign. This action factors through the signature $\varepsilon \colon \mathfrak{S}_2 \times \mathfrak{S}_2 \twoheadrightarrow \mathbb{Z}/2$, so that $\Gamma_*^{\mathfrak{S}_2 \times \mathfrak{S}_2}(\mathbb{Z}) = \Gamma_*^{\mathbb{Z}/2}(\mathbb{Z})$; see §B.1 for the definition of relative lower central series. From Lemma B.4 (whose proof could alternatively be repeated in this situation), we get that for $i \geqslant 2$, we have $\Gamma_i(\mathbb{Z} \rtimes (\mathfrak{S}_2)^2) = 2^{i-1}\mathbb{Z}$, so the lower central series of $\mathbf{B}_{2,2}/\tilde{H} = G/H$ does not stop. $\qquad\square$

6.5.5 Partitioned braids on the projective plane

We use the notation of §A.2 for braids on non-orientable surfaces, with one slight modification: since there is only one crosscap, we denote $c_1^{(\alpha)}$ by γ_α.

We now prove the following theorem, which says when the lower central series of $\mathbf{B}_\lambda(\mathbb{P}^2)$ stops, except for $\mathbf{B}_{2,m}(\mathbb{P}^2)$ with $m \geqslant 3$ (for this case, see Conjecture 6.71):

Theorem 6.54 *Let $n \geqslant 1$ be an integer, let $\lambda = (n_1, ..., n_l)$ be a partition of n. The lower central series of* $\mathbf{B}_\lambda(\mathbb{P}^2)$:

- *stops at Γ_2 if $l = 1$, except for $\mathbf{B}_2(\mathbb{P}^2)$, which is 3-nilpotent.*
- *stops at Γ_3 if $l \geqslant 2$ and $n_i \geqslant 3$ for all i.*
- *does not stop in all the other cases where $l \geqslant 3$.*

Moreover, the lower central series of $\mathbf{B}_{1,m}(\mathbb{P}^2)$ stops at $\Gamma_{v_2(m)+2}$ or at $\Gamma_{v_2(m)+3}$ if $m \geqslant 3$, does not stop if $m = 2$ and stops at Γ_3 if $m = 1$. The lower central series of $\mathbf{B}_{2,2}(\mathbb{P}^2)$ does not stop.

Proof. The first two statements are part of the general results of Corollary 6.29 and Theorem 6.30, except for $\mathbf{B}_1(\mathbb{P}^2) \cong \mathbb{Z}/2$ and $\mathbf{B}_2(\mathbb{P}^2)$, which is the dicyclic group of order 16 (Corollary 6.59). The third statement combines Propositions 6.56 (if λ has blocks of size 1) and 6.60 (if λ has blocks of size 2). The fourth statement combines Propositions 6.57 and 6.65 and the fact that $\mathbf{B}_{1,1}(\mathbb{P}^2) = \mathbf{P}_2(\mathbb{P}^2)$ is the quaternion group Q_8 (Corollary 6.59). The final statement is Proposition 6.68. $\qquad\square$

Blocks of size 1**.** In order to study partitioned braids with blocks of size 1 on the projective plane, we need to know what the Fadell-Neuwirth exact sequence becomes in the exceptional case $n = 1$ of Proposition 6.25.

Proposition 6.55 *For any partition μ of an integer m, we have a short exact sequence:*

$$\mathbf{B}_\mu(\mathbb{M}^2)/\xi^2 \lhook\joinrel\longrightarrow \mathbf{B}_{1,\mu}(\mathbb{P}^2) \longrightarrow\!\!\!\!\!\rightarrow \pi_1(\mathbb{P}^2) \cong \mathbb{Z}/2,$$

where ξ is the central element of $\mathbf{B}_\mu(\mathbb{M}^2)$ given by all the strands going once along $\partial\mathbb{M}^2$.

Sketch of proof. One needs to show that ξ^2 is the image of a generator of $\pi_2(\mathbb{P}^2)$ by the connecting morphism in the long exact sequence of the proof of Proposition 6.25. A generator of $\pi_2(\mathbb{P}^2)$ is given by the canonical projection $\mathbb{S}^2 \twoheadrightarrow \mathbb{P}^2$, which can be lifted to a map from the disk $\mathbb{D}$ to $F_{m+1}(\mathbb{P}^2)$ as follows. Let π denote the projection of $\mathbb{D}$ onto $\mathbb{S}^2 = \mathbb{D}/\partial\mathbb{D}$ sending $\partial\mathbb{D}$ to the south pole $P = -N$. Let $\rho\colon \mathbb{D} \to SO_3(\mathbb{R})$ be the unique continuous function sending each $x \in \mathbb{D}^{\mathrm{o}}$ to the rotation of axis $N \times \pi(x)$ and angle $(N, \pi(x))$ (so that $\rho(x)(N) = \pi(x)$). Let $\mathrm{ev}\colon SO_3(\mathbb{R}) \to F_{m+1}(\mathbb{P}^2)$ be evaluation at a base configuration whose first element is $\pm N$. Then $\mathrm{ev} \circ \rho$ is the required lift. Moreover, when x goes once around $\partial\mathbb{D}$, then $\rho(x)$ goes twice around the circle of rotations of angle π with axis orthogonal to N. Since $\mathbb{P}^2$ is to be thought of as the quotient of the Möbius strip by its boundary (which goes to $\pm N$), this circle of rotations evaluates to the element ξ described in the statement. $\qquad\square$

Proposition 6.56 *Let μ be a partition having at least two blocks. Then the lower central series of $\mathbf{B}_{1,\mu}(\mathbb{P}^2)$ does not stop.*

Proof. We use the extension from Proposition 6.55:

$$\mathbf{B}_\mu(\mathbb{M}^2)/\xi^2 \lhook\joinrel\longrightarrow \mathbf{B}_{1,\mu}(\mathbb{P}^2) \longrightarrow\!\!\!\!\!\to \pi_1(\mathbb{P}^2) \cong \mathbb{Z}/2.$$

We denote the partition $\lambda := (1, \mu)$ of the integer n by $(n_1, ..., n_l)$, and we denote by l' the number of indices i such that $n_i \geqslant 2$. Using notations from the §A.2 (changed to $\gamma_i := c_1^{(i)}$), we have that the quotient in the previous extension is generated by the class of γ_1, and we can write ξ as the following product of commuting braids:

$$\xi = \prod_{i=2}^n \gamma_i^2 (A_{i,i+1} \cdots A_{i,n})^{-1}.$$

Let us consider the quotient G of $\mathbf{B}_{1,\mu}(\mathbb{P}^2)$ by $\Gamma_2(\mathbf{B}_\mu(\mathbb{M}^2)/\xi^2)$. It is an extension of $\mathbb{Z}/2$ by $\mathbf{B}_\mu(\mathbb{M}^2)^{\mathrm{ab}}/2\overline{\xi}$. We recall from Proposition 6.26 that:

$$\mathbf{B}_\mu(\mathbb{M}^2)^{\mathrm{ab}} \cong \mathbb{Z}^{l-1} \times (\mathbb{Z}/2)^{l'}.$$

A basis of the first factor is given by $c_2, ..., c_l$, where each c_i is the common class of the γ_α with α in the i-th block of λ. A $(\mathbb{Z}/2)$-basis of the second factor is given by the elements s_i described in Proposition 3.4. The images of the s_i in G commute with the class c_1 of γ_1, since they have lifts with disjoint support. As a consequence, they are central not only in $\mathbf{B}_\mu(\mathbb{M}^2)^{\mathrm{ab}}/\overline{\xi}$, but in G. Let them generate the subgroup A of G, and consider G/A. There is an extension:

$$\mathbb{Z}^{l-1}/2\overline{\xi} \lhook\joinrel\longrightarrow G/A \longrightarrow\!\!\!\!\!\to \mathbb{Z}/2,$$

where $\overline{\xi} = 2\sum_{i\geqslant 2} n_i c_i$. This extension is not split, but we can quotient further to get a split extension of abelian groups. Namely, we need to kill the element c_1^2 of G. In order to understand it, we use the relations from Corollary A.22: $\gamma_\alpha^2 = A_{\alpha 1} \cdots A_{\alpha n}$ in $\mathbf{B}_{1,\mu}(\mathbb{P}^2)$. In G/A, where the $A_{\alpha\beta}$ are killed if $\alpha, \beta \geqslant 2$, these relations give $\overline{A_{1\alpha}} = \overline{\gamma_\alpha}^2$ if $\alpha \geqslant 2$, and

$$c_1^2 = \overline{\gamma_1}^2 = \overline{A_{12}} \cdots \overline{A_{1n}} = \overline{\gamma_2}^2 \cdots \overline{\gamma_n}^2 = 2\sum_{i\geqslant 2} n_i c_i = \overline{\xi}.$$

In particular, $\overline{\xi}$ commutes with c_1, which implies that it is central in G/A. The quotient H of G/A by $\overline{\xi}$ is thus a semi-direct product:

$$H \cong (\mathbb{Z}^{l-1}/\overline{\xi}) \rtimes \mathbb{Z}/2.$$

Finally, we need to compute the action of c_1 by conjugation on the other c_i. We recall that $[\gamma_\alpha, \gamma_1^{-1}] = A_{1,\alpha}$ in $\mathbf{B}_{1,\mu}(\mathbb{P}^2)$ by Proposition 6.55 and using the presentation of Lemma 6.44, and $\overline{A_{1,\alpha}} = \overline{\gamma_\alpha}^2$ in H, so that $[c_i, c_1^{-1}] = c_i^2$ in H, whence $c_1^{-1} c_i^{-1} c_1 = c_i$, which implies that c_1 acts via $-id$ on $\langle c_2, ..., c_l \rangle = \mathbb{Z}^{l-1}/\overline{\xi} \cong \mathbb{Z}^{l-2} \times (\mathbb{Z}/2)$. Since we have assumed that $l \geqslant 3$, the lower central series of $H = (\mathbb{Z}^{l-2} \times (\mathbb{Z}/2)) \rtimes (\mathbb{Z}/2)$ does not stop; see Corollary B.6. $\qquad\square$

Proposition 6.57 *For $m \geqslant 3$, the lower central series of $\mathbf{B}_{1,m}(\mathbb{P}^2)$ stops either at $\Gamma_{v_2(m)+2}$ or at $\Gamma_{v_2(m)+3}$.*

Proof. Again, we use the extension from Proposition 6.55:

$$\mathbf{B}_m(\mathbb{M}^2)/\xi^2 \longhookrightarrow \mathbf{B}_{1,m}(\mathbb{P}^2) \longrightarrow\!\!\!\!\rightarrow \pi_1(\mathbb{P}^2) \cong \mathbb{Z}/2.$$

We recall that, since $m \geqslant 3$, the lower central series of $\mathbf{B}_m(\mathbb{M}^2)$ stops at Γ_2; see Theorem 6.19. Moreover, $\mathbf{B}_m(\mathbb{M}^2)/\Gamma_\infty = \mathbf{B}_m(\mathbb{M}^2)^{\mathrm{ab}} \cong \mathbb{Z} \times \mathbb{Z}/2$, where the first factor is generated by the common class γ of $\gamma_2, ..., \gamma_{m+1}$ (where $\gamma_i := c_1^{(i)}$), and the second factor is generated by the common class σ of the σ_i; see Proposition 6.11. The image of the central element ξ of $\mathbf{B}_m(\mathbb{M}^2)^{\mathrm{ab}}$ is then:

$$\overline{\xi} = \sum_{i=2}^{m+1} \overline{\gamma_i^2(A_{i,i+1}\cdots A_{i,m+1})^{-1}} = 2m\gamma - 2 \cdot \frac{m(m+1)}{2}\sigma = 2m\gamma,$$

so that $\mathbf{B}_m(\mathbb{M}^2)/(\xi^2 \cdot \Gamma_2) \cong \mathbb{Z}/4m \times \mathbb{Z}/2$. Let us consider the quotient G of $\mathbf{B}_{1,m}(\mathbb{P}^2)$ by the image of $\Gamma_2(\mathbf{B}_m(\mathbb{M}^2))$. We recall that the image of $\Gamma_2(\mathbf{B}_m(\mathbb{M}^2)) = \Gamma_\infty(\mathbf{B}_m(\mathbb{M}^2))$ is inside $\Gamma_\infty(\mathbf{B}_{1,m}(\mathbb{P}^2))$, so that G and $\mathbf{B}_{1,m}(\mathbb{P}^2)$ have the same associated Lie ring. Now, G is an extension:

$$\mathbb{Z}/4m \times \mathbb{Z}/2 \longhookrightarrow G \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}/2.$$

We can already see that G is finite, and deduce that its lower central series stops, so that $\Gamma_*(\mathbf{B}_{1,m}(\mathbb{P}^2))$ stops too. In order to be more precise, let us recall that $G = \langle \gamma, \sigma, \gamma_1 \rangle$, where the class of γ_1 generates the quotient in the previous extension. We note that γ_1^2 is in the kernel, so it commutes with γ and σ. Since it obviously commutes with γ_1, it is central in G. Moreover, as in the previous proof, the boundary relations in $\mathbf{B}_{1,m}(\mathbb{P}^2)$ give:

$$\gamma = \overline{\gamma_i^2} = \overline{A_{i,1}\cdots A_{i,m+1}} = \overline{A_{i,1}} \text{ for } i \geqslant 2, \quad \text{and} \quad \gamma_1^2 = \overline{A_{1,2}\cdots A_{1,m+1}} = \gamma^{2m} = \overline{\xi}.$$

Thus, the quotient of G by its central subgroup $A = \langle \gamma^{2m} \rangle$ is an extension of $\mathbb{Z}/2$ by $(\mathbb{Z}/2m) \times \mathbb{Z}/2$. Since $\gamma_1^2 = 1$ in G/A, this extension splits as a semi-direct product:

$$G/A \cong ((\mathbb{Z}/2m) \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2.$$

Finally, γ_1 commutes with σ, and

$$[\gamma, \gamma_1^{-1}] = \overline{[\gamma_2, \gamma_1^{-1}]} = \overline{A_{12}} = \gamma^2,$$

which implies that $\gamma_1^{-1}\gamma^{-1}\gamma_1 = \gamma$. This means that $\mathbb{Z}/2$ acts trivially on the $\mathbb{Z}/2$ factor, and via $-id$ on the $\mathbb{Z}/2m$ factor. Finally, we can compute completely the lower central series of $G/A \cong \mathbb{Z}/2 \times (\mathbb{Z}/2m \rtimes \mathbb{Z}/2)$ using Corollary B.8. Precisely, $\Gamma_i(G/A) = 2^{i-1}\mathbb{Z}/(2m)$ for $i \geqslant 2$. Hence it stops at Γ_3 if m is odd and, in general, it stops at $\Gamma_{v_2(m)+2}$. Now, we can use Corollary 1.3 to deduce that the lower central series of G (whence the one of $\mathbf{B}_{1,m}(\mathbb{P}^2)$) stops at $\Gamma_{v_2(m)+2}$ or at $\Gamma_{v_2(m)+3}$. $\qquad\square$

Remark 6.58 Similarly to the situation of $\mathbf{B}_{2,m}(\mathbb{S}^2)$ (see Remark 6.52), it seems difficult to decide theoretically between the two possibilities in Proposition 6.57. However, based on experimental calculations using GAP [GAP21] and the package NQ [Nic96], we conjecture that the lower central series of $\mathbf{B}_{1,m}(\mathbb{P}^2)$ always stops at $\Gamma_{v_2(m)+3}$.

We are left with two cases to consider where there is a block of size 1, namely $\mathbf{B}_{1,1}(\mathbb{P}^2)$ and $\mathbf{B}_{1,2}(\mathbb{P}^2)$. The group $\mathbf{B}_{1,1}(\mathbb{P}^2) = \mathbf{P}_2(\mathbb{P}^2)$ is isomorphic to the quaternion group Q_8 (see Corollary 6.59 below), which is 2-nilpotent, so its lower central series stops at Γ_3. Notice that this means that the conclusion of Proposition 6.57 is correct also for $m = 1$, since $v_2(1) = 0$, so that proposition would assert that the lower central series of $\mathbf{B}_{1,1}(\mathbb{P}^2)$ stops at Γ_2 or Γ_3. In contrast, we will show that the lower central series of $\mathbf{B}_{1,2}(\mathbb{P}^2)$ does not stop (Proposition 6.65). The study of this latter case is postponed, and will be part of our study of $\mathbf{B}_{2,m}(\mathbb{P}^2)$, of which a presentation will be computed in Proposition 6.61 for every $m \geqslant 1$.

Blocks of size 2. We now study the case where there is at least one block with exactly two strands.

Proposition 6.55 can be used to recover the following classical calculations from [Bus66, p. 87]:

Corollary 6.59 *The pure braid group* $\mathbf{P}_2(\mathbb{P}^2)$ *is isomorphic to the quaternion group* Q_8 *(which is 2-nilpotent), and* $\mathbf{B}_2(\mathbb{P}^2)$ *to the dicyclic group of order* 16 *(which is 3-nilpotent). Precisely, a presentation of the latter is:*

$$\mathbf{B}_2(\mathbb{P}^2) = \left\langle \sigma_1, \gamma_1 \mid \gamma_1^2 = [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2 \right\rangle.$$

Proof. Recall that the dicyclic group of order 16 can be defined by the presentation:

$$Dic_{16} = \left\langle s, x, y \mid sx = ys, \ x^2 = y^2 = (xy)^2 = s^2 \right\rangle,$$

and that it contains the subgroup Q_8 of quaternions as the index-2 subgroup generated by x and y. Notice that, modulo the other relations, $(xy)^2 = s^2$ is equivalent to $s^2 x^{-1} y x s^2 = s^2$, which is equivalent to $[y, x^{-1}] = s^2$ by passing to the inverses. We can also use the first relation to eliminate $y = sxs^{-1}$. As a consequence, we also have:

$$Dic_{16} = \left\langle s, x \mid x^2 = [sxs^{-1}, x^{-1}] = s^2 \right\rangle.$$

It is easy to check that the elements σ_1, γ_1 and γ_2 of $\mathbf{B}_2(\mathbb{P}^2)$ satisfy the above relations, so that $s \mapsto \sigma_1$ and $x \mapsto \gamma_1$ define a morphism φ from Dic_{16} to $\mathbf{B}_2(\mathbb{P}^2)$. Indeed, the relation $\gamma_1^2 = \sigma_1^2$ is one of the boundary relations from Corollary A.18, and an isotopy witnessing the last one is drawn in Figure 6.1. Note that φ sends the element $y = sxs^{-1}$ to $\sigma_1\gamma_1\sigma_1^{-1} = \gamma_2$.

By taking $m = n = 1$, the short exact sequence of Proposition 6.55 specialises to:

$$\pi_1(\mathbb{M}^2)/\xi^2 \cong \mathbb{Z}/4 \longhookrightarrow \mathbf{P}_2(\mathbb{P}^2) \longtwoheadrightarrow \pi_1(\mathbb{P}^2) \cong \mathbb{Z}/2.$$

Indeed, $B_1(\mathbb{M}^2) = \pi_1(\mathbb{M}^2)$ is isomorphic to $\mathbb{Z}$, and the element ξ, which is a loop parallel to the boundary of the Möbius strip, is the square of a generator. From this, we deduce that $\mathbf{P}_2(\mathbb{P}^2)$ has eight elements, and that it is generated by γ_1 (the image of a generator of $\pi_1(\mathbb{M}^2)$) and γ_2 (a lift of the generator of $\pi_1(\mathbb{P}^2)$). As a consequence, φ must induce an isomorphism between $Q_8 = \langle x, y \rangle \subset Dic_{16}$ and $\mathbf{P}_2(\mathbb{P}^2)$. Then, we can use the usual extension

$$\mathbf{P}_2(\mathbb{P}^2) \longhookrightarrow \mathbf{B}_2(\mathbb{P}^2) \longtwoheadrightarrow \mathfrak{S}_2$$

to deduce that σ_1, γ_1 and γ_2 generate $\mathbf{B}_2(\mathbb{P}^2)$, and that $\mathbf{B}_2(\mathbb{P}^2)$ has sixteen elements. Hence φ is an isomorphism. □

We first deal with the case where there are at least three blocks:

Proposition 6.60 *Let μ be a partition having at least two blocks. Then the lower central series of* $\mathbf{B}_{2,\mu}(\mathbb{P}^2)$ *does not stop.*

Proof. We are looking for a quotient whose lower central series can be computed, and does not stop. Let us consider the Fadell-Neuwirth extension from Proposition 6.25:

$$\mathbf{B}_\mu(\mathbb{M}^2 - \{pt\}) \longhookrightarrow \mathbf{B}_{2,\mu}(\mathbb{P}^2) \longtwoheadrightarrow \mathbf{B}_2(\mathbb{P}^2).$$

We use notations similar to the ones from the proof of Proposition 6.56: we denote the partition $\lambda := (1, 1, \mu)$ of the integer n by $(n_1, ..., n_l)$, and we denote by l' the number of indices i such that $n_i \geqslant 2$. We also use notations from §A.2, changed to $\gamma_\alpha := c_1^{(\alpha)}$.

In order to get a more manageable extension, we first take the quotient G of $\mathbf{B}_{2,\mu}(\mathbb{P}^2)$ by $\Gamma_2(\mathbf{B}_\mu(\mathbb{M}^2 - \{pt\}))$, getting an extension:

$$\mathbf{B}_\mu(\mathbb{M}^2 - \{pt\})^{\mathrm{ab}} \longhookrightarrow G \longtwoheadrightarrow \mathbf{B}_2(\mathbb{P}^2).$$

The kernel $\mathbf{B}_\mu(\mathbb{M}^2 - \{pt\})^{\mathrm{ab}}$ is computed in Proposition 6.26: it is $(\pi_1(\mathbb{M}^2 - \{pt\})^{\mathrm{ab}})^{l-2} \times (\mathbb{Z}/2)^{l'}$, where the first factor is generated by the classes c_i of γ_α, a_{1i} of $A_{1\alpha}$ and a_{2i} of $A_{2\alpha}$ (for α in the i-th block, with $i \geqslant 3$), subject to the relations $a_{1i} + a_{2i} = 2c_i$ (for each i, we get a copy of $\pi_1(\mathbb{M}^2 - \{pt\})^{\mathrm{ab}} \cong \mathbb{Z}^2$), and the second one is generated by the classes s_i of the σ_α (for α and $\alpha+1$ in the i-th block of λ, which is possible if $n_i \geqslant 2$). The group G is generated by these generators, together with the classes of σ_1 and γ_1 (whose images generate $\mathbf{B}_2(\mathbb{P}^2)$).

The elements s_i commute with the other elements of $\mathbf{B}_\mu(\mathbb{M}^2 - \{pt\})^{\mathrm{ab}}$ (which is abelian), but also with σ_1 and γ_1 (for reasons of support). Hence they are central elements of G. Let us denote by $A \cong (\mathbb{Z}/2)^{l'}$ the central subgroup they generate, and let us consider the extension:

$$\mathbb{Z}^{2(l-2)} \longhookrightarrow G/A \longtwoheadrightarrow \mathbf{B}_2(\mathbb{P}^2).$$

Corollary 6.59 gives a presentation of the quotient, namely:

$$\mathbf{B}_2(\mathbb{P}^2) = \left\langle \sigma_1, \gamma_1 \mid \gamma_1^2 = [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2 \right\rangle.$$

We now compute a presentation of G/A, using the tools from §A.3. Generators are c_i, a_{1i} and a_{2i} (for $3 \leqslant i \leqslant l$), together with σ_1 and γ_1. Relations defining the kernel are the ones saying that the c_i, the a_{1i} and the a_{2i} commute with each other, together with $a_{1i} + a_{2i} = 2c_i$ (for each $i \geqslant 3$). The latter could be used to eliminate $a_{2i} = 2c_i - a_{1i}$. However, we will choose not to do so here, and to give a redundant, but more tractable presentation of G/A. Relations lifting the above presentation of the quotient are:

$$\begin{cases} \gamma_1^2 = \sigma_1^2 \cdot a_{13}^{n_3} \cdots a_{1l}^{n_l}, \\ [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2. \end{cases}$$

Indeed, these hold in $\mathbf{B}_{2,\mu}(\mathbb{P}^2) \subset \mathbf{B}_n(\mathbb{P}^2)$: the first one is one of the boundary relations from Corollary A.18, and the second one is the one pictured in Figure 6.1. Moreover, these are clearly lifts of the relations defining $\mathbf{B}_2(\mathbb{P}^2)$.

We are left with understanding how σ_1 and γ_1 (whence also $\gamma_2 = \sigma_1\gamma_1\sigma_1^{-1}$) act by conjugation on the c_i, the a_{1i} and the a_{2i}. We claim that the following relations hold in G/A:

$$\begin{cases} \sigma_1 a_{1i} = a_{2i}\sigma_1, \quad \sigma_1 a_{2i} = a_{1i}\sigma_1 \ \text{ and } \ \sigma_1 \rightleftarrows c_i \\ \gamma_1 a_{ki}\gamma_1^{-1} = (-1)^{\delta_{1k}} a_{ki} \ \text{ and } \ \gamma_1 c_i \gamma_1^{-1} = c_i - a_{1i}, \end{cases}$$

where we use additive notations in the (abelian) subgroup generated by the c_i, the a_{1i} and the a_{2i}. These are images of relations holding in $\mathbf{B}_{2,\mu}(\mathbb{P}^2)$, which can be proved by drawing explicit isotopies. Precisely, the first one comes from $\sigma_1 A_{1i}\sigma_1^{-1} = A_{2i}$, the second one from $\sigma_1 A_{2i}\sigma_1^{-1} = A_{2i}^{-1}A_{1i}A_{2i}$, and the third one from the commutation of σ_1 with all the γ_α if $\alpha \geqslant 3$. The other relation comes from γ_1 commuting with A_{2i}, and from $[\gamma_i, \gamma_1^{-1}] = A_{1i}$ (Figure 6.1), that is, $\gamma_1^{-1}\gamma_i\gamma_1 = A_{1i}^{-1}\gamma_i$. Notice that the relation involving $\gamma_1 a_{1i}\gamma_1^{-1}$ can be deduced from the other two, using $a_{1i} = 2c_i - a_{2i}$.

We now have a presentation of G/A. In order to get a simpler quotient, we quotient further by σ_1^2 and γ_1^2. That is, we add the relations $\sigma_1^2 = \gamma_1^2 = 1$ to the previous presentation. The result is a split extension:

$$\mathbb{Z}^{2(l-2)} \Big/ \left(\sum_{i=3}^l n_i a_{1i} = \sum_{i=3}^l n_i a_{2i} = 0 \right) \rtimes W_2.$$

If we quotient further by $\sum n_i c_i$ (which is central in the above semi-direct product, since it is fixed by the action of W_2), we obtain a semi-direct product $M \rtimes W_2$, where M has an explicit workable description. Precisely, as a W_2-representation, $M \cong \Lambda \otimes B$, where Λ is the canonical representation of W_2 defined in §B.2.4, and B is the quotient of $\mathbb{Z}^{l-2}$ by the vector $(n_3, ..., n_l)$, seen as a trivial W_2-representation. Precisely, if e_i $(i = 3, \ldots, l)$ is the generating family of B obtained from the canonical basis of $\mathbb{Z}^2$, an isomorphism $\Lambda \otimes B \cong M$ is given, with the notations from Remark B.15, by $a \otimes e_i \mapsto a_{2i}$, $b \otimes e_i \mapsto a_{1i}$ and $c \otimes e_i \mapsto c_i$.

We finally use our hypothesis: since $l \geqslant 4$, the rank of B is not 0, so it surjects onto $\mathbb{Z}$. Thus, M surjects onto $\Lambda \otimes \mathbb{Z} = \Lambda$ (as a W_2-representation), and $M \rtimes W_2$ surjects onto $\Lambda \rtimes W_2$, whose lower central series (computed in Proposition B.12) does not stop. $\qquad\square$

Now, we are left with the case where there are precisely two blocks, one of which has exactly two strands. First of all, we give an explicit presentation of the associated group:

Proposition 6.61 *Let $m \geqslant 1$ be an integer. The group $\mathbf{B}_{2,m}(\mathbb{P}^2)$ admits the presentation with generators $\sigma_1, \sigma_3, \sigma_4, ..., \sigma_{m+1}, \gamma_1, \gamma_3$ and A_{23}, subject to the following relations:*

$$
\begin{cases}
(\mathbb{P}R1) & \sigma_4, ..., \sigma_{m+1} \; \rightleftarrows \; \gamma_3, A_{23}; \\
(\mathbb{P}R2) & A_{23} \; \rightleftarrows \; \sigma_3 \gamma_3 \sigma_3^{-1}; \\
(\mathbb{P}R3) & (\sigma_3 A_{23})^2 = (A_{23}\sigma_3)^2; \\
(\mathbb{P}R4) & [\sigma_i \gamma_i \sigma_i^{-1}, \gamma_i^{-1}] = \sigma_i^2 \;\; \text{for } i \in \{1,3\}; \\
(\mathbb{P}R5) & \gamma_1^2 = \sigma_1 \left(\prod_{k=2}^{m+1} (\sigma_k \cdots \sigma_3) A_{23} (\sigma_k \cdots \sigma_3)^{-1} \right) \sigma_1; \\
(\mathbb{P}R6) & \sigma_1 \; \rightleftarrows \; \sigma_3, \sigma_4, ..., \sigma_{m+1}, \gamma_3; \\
(\mathbb{P}R7) & \gamma_1 \; \rightleftarrows \; \sigma_3, \sigma_4, ..., \sigma_{m+1}, A_{23}; \\
(\mathbb{P}R8) & \gamma_3^2 = (\sigma_1^{-1} A_{23} \sigma_1) A_{23} \cdot \prod_{k=3}^{m+1} (\sigma_k \cdots \sigma_4) \sigma_3^2 (\sigma_k \cdots \sigma_4)^{-1}; \\
(\mathbb{P}R9) & [\gamma_3, \gamma_1^{-1}] = \sigma_1^{-1} A_{23} \sigma_1.
\end{cases}
$$

Remark 6.62 This set of generators is not minimal, although it is close to being minimal. Indeed, we can eliminate A_{23} using $(\mathbb{P}R9)$. The set of generators thus obtained is then minimal, at least for $m = 1, 2$, since the classes of the generators are a $\mathbb{Z}/2$-basis of the abelianisation in this case. However, this elimination would render the relations much less tractable; the above seems a better compromise between the number of generators and readability of the relations.

Proof of Proposition 6.61. Let us begin by checking that these relations hold for the usual elements $\sigma_1, \sigma_3, \sigma_4, ..., \sigma_{m+1}, \gamma_1, \gamma_3$ and A_{23} of $\mathbf{B}_{2,m}(\mathbb{P}^2)$. Let us first remark that we can express other usual elements in terms of these elements. Namely, $A_{12} = \sigma_1^{-1} A_{23} \sigma_1$ and, if $k \in \{3, \ldots, m+2\}$, γ_k (resp. A_{2k}, A_{1k}) is obtained from γ_3 (resp. A_{23}, A_{13}) by conjugation by $\sigma_k \cdots \sigma_4$ on the left (by convention, this product equals 1 if $k = 3$). Then one can see that $(\mathbb{P}R5)$ and $(\mathbb{P}R8)$ are the boundary relations from Corollary A.18, corresponding to the strands 1 and 3 (recall that $A_{12} = \sigma_1^2$ and that σ_1 commutes with $\sigma_3, ..., \sigma_{m+1}$). All the other relations can be checked by drawing explicit isotopies. Precisely, the relations $(\mathbb{P}R1)$, $(\mathbb{P}R2)$, $(\mathbb{P}R6)$ and $(\mathbb{P}R7)$ are commutation relations between elements having "disjoint support"; $(\mathbb{P}R4)$ and $(\mathbb{P}R9)$ are instances of the relation drawn in Figure 6.1; $(\mathbb{P}R3)$ can either be proved by drawing an explicit isotopy, or it can be deduced from the similar relation in $\mathbf{B}_{1,2}$ (see Lemma 3.15), by considering the morphism from $\mathbf{B}_{1,2}$ to $\mathbf{B}_{2,m}(\mathbb{P}^2)$ induced by a well-chosen embedding of $\mathbb{D}$ into $\mathbb{P}^2$.

In order to show that these relations describe the group, we now apply the methods of §A.3 to the Fadell-Neuwirth extension from Proposition 6.25:

$$
\mathbf{B}_m(\mathbb{P}^2 - \{2 \text{ pts}\}) \lhook\joinrel\longrightarrow \mathbf{B}_{2,m}(\mathbb{P}^2) \longrightarrow\joinrel\twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2).
$$

Since $\mathbb{P}^2 - \{2 \text{ pts}\} = \mathcal{N}_{1,2}$, a presentation of the kernel is given by Proposition A.16, for $n = 1$. In order for it to identify with the right subgroup of $\mathbf{B}_{2,m}(\mathbb{P}^2)$ (corresponding to braids on the strands $3, \ldots, m+2$), indices are shifted by 2 (so that, for instance, c_1 becomes γ_3), and we take $z_1 = A_{23}$; this last choice changes the presentation a little bit, but one can easily figure out the necessary modifications. Thus, $(BN1)$ and (BN_4) give $(\mathbb{P}R1)$; $(BN2)$ and $(BN6)$ are empty; $(BN3)$ is the case $i = 3$ of $(\mathbb{P}R4)$; $(BN5)$ is $(\mathbb{P}R2)$; finally, $(BN7)$ is the case $i = 3$ of $(\mathbb{P}R3)$.

A presentation of the quotient is given by Corollary 6.59 and its proof:

$$
\mathbf{B}_2(\mathbb{P}^2) = \left\langle \sigma_1, \gamma_1, \gamma_2 \;\big|\; \sigma_1 \gamma_1 = \gamma_2 \sigma_1, \; \gamma_1^2 = \gamma_2^2 = [\gamma_2, \gamma_1^{-1}] = \sigma_1^2 \right\rangle.
$$

Notice that we can eliminate $\gamma_2 = \sigma_1\gamma_1\sigma_1^{-1}$ to get:

$$\mathbf{B}_2(\mathbb{P}^2) = \left\langle \sigma_1, \gamma_1, \gamma_2 \mid \gamma_1^2 = [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2 \right\rangle.$$

The elements $\gamma_1, \sigma_1 \in \mathbf{B}_{2,m}(\mathbb{P}^2)$ are lifts of the elements $\gamma_1, \sigma_1 \in \mathbf{B}_2(\mathbb{P}^2)$. The second relation holds without change for these lifts, giving the case $i = 1$ of $(\mathbb{P}R4)$. The relation $\gamma_1^2 = \sigma_1^2$ lifts to the boundary relation associated with the first strand, which is $(\mathbb{P}R5)$.

We are left with finding relations describing the action of σ_1 and γ_1 (or of σ^{-1} and γ^{-1}) by conjugation on the other generators. The commutation relations $(\mathbb{P}R6)$ and $(\mathbb{P}R7)$ are describing most of it. Only two elements still need to be expressed in terms of the generators $\sigma_3, \ldots, \sigma_{m+1}$, γ_3 and A_{23} of the kernel, namely $\sigma_1^{-1}A_{23}\sigma_1$ and $\gamma_1^{-1}\gamma_3\gamma_1$. The boundary relation $(\mathbb{P}R8)$ deals with $\sigma_1^{-1}A_{23}\sigma_1$. Finally, $(\mathbb{P}R9)$ deals with $\gamma_1^{-1}\gamma_3\gamma_1$: it says that $\gamma_1^{-1}\gamma_3\gamma_1 = \gamma_3\sigma_1^{-1}A_{23}\sigma_1$, which is equivalent to an expression of $\sigma_1 A_{23}\sigma_1^{-1}$ in terms of the generators of the kernel, modulo the previous relation $(\mathbb{P}R8)$. This finishes the proof that the above relations are the ones obtained using the method of §A.3, whence the result. $\qquad\square$

Let us now consider the case $m = 1$. In this case, there is no σ_i for $i \geqslant 3$, so the presentation is much simpler; in fact, in the extension of the proof, the kernel is just $\pi_1(\mathbb{P}^2 - \{2 \text{ pts}\})$, which is free on γ_3 and A_{23}. Thus, $(\mathbb{P}R1)$, $(\mathbb{P}R2)$ and $(\mathbb{P}R3)$ are empty in this case, and there is no case $i = 3$ in $(\mathbb{P}R4)$. The other relations reduce to:

Corollary 6.63 *The group* $\mathbf{B}_{2,1}(\mathbb{P}^2)$ *has a presentation with 4 generators* σ_1, γ_1, γ_3 *and* A_{23} *and 6 relations (indexed as above):*

$$
\begin{cases}
(4) & [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2; \\
(5) & \gamma_1^2 = \sigma_1 A_{23}\sigma_1; \\
(6) & \sigma_1 \rightleftarrows \gamma_3;
\end{cases}
\qquad
\begin{cases}
(7) & \gamma_1 \rightleftarrows A_{23}; \\
(8) & \gamma_3^2 = \sigma_1^{-1}A_{23}\sigma_1 A_{23}; \\
(9) & [\gamma_3, \gamma_1^{-1}] = \sigma_1^{-1}A_{23}\sigma_1.
\end{cases}
$$

Remark 6.64 This is better than Van Buskirk's presentation from [Bus66, Lem. p. 84], which has 6 generators and 13 relations. Note that in our language, his generators are as follows, where the right-hand side of each equation uses our notation and the left-hand side uses his:

$$
\begin{cases}
\sigma_2 = \sigma_1, \\
a_2 = \sigma_1^{-1}A_{23}\sigma_1, \\
a_3 = A_{23},
\end{cases}
\qquad
\begin{cases}
\rho_1 = \gamma_3, \\
\rho_2 = \gamma_1\sigma_1^{-1}A_{23}^{-1}\sigma_1, \\
\rho_3 = \sigma_1^{-1}\gamma_1\sigma_1.
\end{cases}
$$

Proposition 6.65 *The lower central series of* $\mathbf{B}_{2,1}(\mathbb{P}^2)$ *does not stop.*

Proof. Let G be the quotient of $\mathbf{B}_{1,2}(\mathbb{P}^2)$ by the single relation $\sigma_1\gamma_1\sigma_1^{-1} = \gamma_1^{-1}$. Let us consider the presentation of G given by the presentation of Corollary 6.63, to which this relation is added. Then relation (4) becomes $\sigma_1^2 = 1$, that is, $\sigma_1^{-1} = \sigma_1$. Relation (5) becomes $A_{23} = \sigma_1\gamma_1^2\sigma_1 = \gamma_1^{-2}$, hence (7) becomes redundant. Relation (8) is then equivalent to $\gamma_3^2 = 1$. Relation (9) becomes $\gamma_3\gamma_1^{-1}\gamma_3^{-1} = \gamma_1$. If we add to this relation (6), which says that σ_1 commutes with γ_3, we get a presentation of $\mathbb{Z} \rtimes (\mathbb{Z}/2)^2$ (where both elements of a basis of $(\mathbb{Z}/2)^2$ act via $-id$). Thus, $G \cong \mathbb{Z} \rtimes (\mathbb{Z}/2)^2$, whose lower central series does not stop (one can either compute it with the method of the appendix, or quotient further by $\sigma_1 = \gamma_3$ to get $\mathbb{Z} \rtimes (\mathbb{Z}/2)$ as a quotient, and apply Corollary B.8). $\qquad\square$

Remark 6.66 Even if the imposed relation looks very much like a relation defining the infinite dihedral group $\mathbb{Z} \rtimes (\mathbb{Z}/2)$, it is not at all clear *a priori* why adding this one relation should work. Much experimentation has been needed before ending up here.

Next, let us consider the case $m = 2$. In this case, there is no σ_i for $i \geqslant 4$. Thus, $(\mathbb{P}R1)$ is empty, and the boundary relations $(\mathbb{P}R5)$ and $(\mathbb{P}R8)$ become much simpler. Let us spell out the result in this case:

Corollary 6.67 *The group* $\mathbf{B}_{2,2}(\mathbb{P}^2)$ *has a presentation with* 5 *generators* σ_1, γ_1, γ_3 *and* A_{23} *and* 8 *relations (indexed as above):*

$$\begin{cases} (2) & A_{23} \rightleftarrows \sigma_3\gamma_3\sigma_3^{-1}; \\ (3) & (\sigma_3 A_{23})^2 = (A_{23}\sigma_3)^2; \\ (4) & [\sigma_i\gamma_i\sigma_i^{-1}, \gamma_i^{-1}] = \sigma_i^2 \ for \ i \in \{1,3\}; \\ (5) & \gamma_1^2 = \sigma_1 A_{23}\sigma_3 A_{23}\sigma_3^{-1}\sigma_1; \end{cases} \qquad \begin{cases} (6) & \sigma_1 \rightleftarrows \sigma_3, \gamma_3; \\ (7) & \gamma_1 \rightleftarrows \sigma_3, A_{23}; \\ (8) & \gamma_3^2 = \sigma_1^{-1}A_{23}\sigma_1 A_{23}\sigma_3^2; \\ (9) & [\gamma_3, \gamma_1^{-1}] = \sigma_1^{-1}A_{23}\sigma_1. \end{cases}$$

Proposition 6.68 *The lower central series of* $\mathbf{B}_{2,2}(\mathbb{P}^2)$ *does not stop.*

Proof. Let us consider, as above, the projection $p\colon \mathbf{B}_{2,2}(\mathbb{P}^2) \twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2) \cong Dic_{16}$ induced by forgetting the last two strands. Recall that Corollary 6.59 gives a presentation of this quotient:

$$\mathbf{B}_2(\mathbb{P}^2) = \left\langle \sigma_1, \gamma_1 \ \middle| \ \gamma_1^2 = [\sigma_1\gamma_1\sigma_1^{-1}, \gamma_1^{-1}] = \sigma_1^2 \right\rangle.$$

Since the second relation is already true in $\mathbf{B}_{2,2}(\mathbb{P}^2)$ (it is the case $i = 1$ of relation (4) in Corollary 6.67), the projection p becomes split if we impose the relation $\sigma_1^2 = \gamma_1^2$. We will in fact consider the quotient G of $\mathbf{B}_{2,2}(\mathbb{P}^2)$ by the two relations:

$$\begin{cases} (Q1) & \sigma_1^2 = \gamma_1^2; \\ (Q2) & \sigma_3\gamma_3\sigma_3^{-1} = \gamma_3^{-1} \end{cases}$$

Since the second relator also sits in the kernel of p, we get a split projection $\bar{p}\colon G \twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2)$.

Let us consider the presentation of G given by the presentation of Corollary 6.67, together with the two relations $(Q1)$ and $(Q2)$. Modulo $(Q1)$, relation (5) becomes $\sigma_3 A_{23}\sigma_3^{-1} = A_{23}^{-1}$. Modulo $(Q2)$, relation (2) says that A_{23} commutes with γ_3, and the case $i = 3$ of relation (4) gives $\sigma_3^2 = 1$. At this point, let us remark that the relations obtained so far are saying that the subgroup $\langle \gamma_3, A_{23}, \sigma_3 \rangle$ is a quotient of $\mathbb{Z}^2 \rtimes (\mathbb{Z}/2)$, where the action of $\mathbb{Z}/2$ is by $-id$.

Continuing our investigation, we remark that relation (3) is a consequence of the previous relation (both sides of it are killed modulo these). Relation (8) becomes $\sigma_1^{-1}A_{23}\sigma_1 = \gamma_3^2 A_{23}^{-1}$, and (remembering that $\sigma_3^2 = 1$) relation (9) becomes $\gamma_1^{-1}\gamma_3\gamma_1 = A_{23}\gamma_3^{-1}$. If we add relations (6) and (7) without change, we get a presentation of G, which will allow us to describe it in an explicit way.

Let us first consider the subgroup $A := \langle \gamma_3, A_{23} \rangle$ (which is abelian by relation (2)). Relations $(Q2)$ and (2) imply that it is stable under conjugation by σ_3 (which equals σ^{-1}, by (4)). Relations (6) and (8) imply that it is stable under conjugation by σ_i^{-1}, and also by σ_i (as one sees by conjugating (8) by σ_i, taking (6) into account). In the same way, relations (7) and (9) imply that it is stable under conjugation by $\gamma_1^{\pm 1}$. Finally, all this implies that it is normal in G. Moreover, the presentation of G/A that we get from the one of G clearly gives $G/A \cong Dic_{16} \times (\mathbb{Z}/2)$, and in fact, the relations defining G/A were already true in G for the generators σ_1, γ_1 and σ_3, so $G \twoheadrightarrow G/A$ splits, and finally:

$$G \cong A \rtimes (Dic_{16} \times (\mathbb{Z}/2)).$$

We are left with understanding A (which is a quotient of $\mathbb{Z}^2$) and the action of $K := Dic_{16} \times (\mathbb{Z}/2)$ on it. In order to do this, let us consider the relations describing the action of K on A, namely:

$$\begin{cases} (Q2) & \sigma_3\gamma_3\sigma_3^{-1} = \gamma_3^{-1}; \\ (5) & \sigma_3 A_{23}\sigma_3^{-1} = A_{23}^{-1}; \\ (6) & \sigma_1\gamma_3\sigma_1^{-1} = \gamma_3; \\ (8) & \sigma_1 A_{23}\sigma_1^{-1} = A_{23}^{-1}\gamma_3^2; \\ (9) & \gamma_1\gamma_3\gamma_1^{-1} = \gamma_3^{-1}A_{23}; \\ (7) & \gamma_1 A_{23}\gamma_1^{-1} = A_{23}. \end{cases}$$

We remark that these already define an action of K on $\mathbb{Z}^2$, which is exactly the action on $\mathbb{Z}^2 = \Gamma$ considered at the end of §B.2.4. Precisely, with the identifications $\sigma_1 \mapsto \sigma$, $\gamma_1 \mapsto \gamma$ and $\sigma_3 \mapsto \tau$ for generators of K, we get an equivariant map $\Gamma \twoheadrightarrow A$ sending a to A_{23} and c to γ_3 (with the

notations of Remark B.15). This induces a surjective morphism from $\Gamma \rtimes K$ onto $A \rtimes K = G$. It is then easy to check that all the relations defining G are in fact already true in $\Gamma \rtimes K$, which allows us to define a converse isomorphism $G \cong \mathbb{Z}^2 \rtimes K$.

Finally, Proposition B.12, together with the equality $\Gamma_*^K(\Lambda) = \Gamma_*^{W_2}(\Lambda)$ from the end of §B.2.4, gives us a complete description of the lower central series of G which, in particular, does not stop. $\qquad\square$

Remark 6.69 The projection onto $Dic_{16} \times (\mathbb{Z}/2)$ in the proof can be seen as coming from the geometry. Precisely, it is the factorisation through G of the projection

$$q \colon \mathbf{B}_{2,2}(\mathbb{P}^2) \twoheadrightarrow \mathbf{B}_2(\mathbb{P}^2) \times \mathfrak{S}_2 \cong Dic_{16} \times (\mathbb{Z}/2)$$

whose first factor forgets the last two strands and whose second factor forgets the first two strands and then applies the usual projection $\pi \colon \mathbf{B}_2(\mathbb{P}^2) \twoheadrightarrow \mathfrak{S}_2$.

Remark 6.70 This quotient looks very much like the one from the proof of Proposition 6.65, and the same remark applies (see Remark 6.66). Namely, $(Q1)$ is a natural relation to impose (making the extension split), whereas it is much less clear why quotienting by $(Q2)$ (which is the same relation as in the aforementioned proof, up to re-indexing the strands) should work.

The remaining cases to consider are $\mathbf{B}_{2,m}(\mathbb{P}^2)$ for $m \geqslant 3$. These are the only examples of partitioned surface braid groups for which we have not been able to answer the question of whether their lower central series stop. We can still say something about these lower central series, using the proof of Proposition 6.60. Recall that in this proof, the hypothesis on the number of blocks of μ was not used until the end. Moreover, in the case $\mu = (m)$ and $m \geqslant 3$, Proposition 6.19 applies, implying that the first quotient G is the quotient by $\Gamma_\infty(\mathbf{B}_m(\mathbb{M}^2 - \{pt\}))$. The latter must be contained in $\Gamma_\infty(\mathbf{B}_{2,m}(\mathbb{P}^2))$ so, in order to understand the lower central series of $\mathbf{B}_{2,m}(\mathbb{P}^2)$, we only need to understand the one of G. Then, the central subgroup $A = \langle s_3 \rangle \cong \mathbb{Z}/2$ (where s_3 is the class of σ_3) injects into $G^{\mathrm{ab}} = \mathbf{B}_{2,m}(\mathbb{P}^2)^{\mathrm{ab}}$ (which was computed in Proposition 6.26), so we can apply Corollary 1.3 to see that the lower central series of G stops if and only if the one of G/A does. We have the same presentation of G/A as in the proof of Proposition 6.60, from its decomposition as an extension of $\mathbf{B}_2(\mathbb{P}^2) = Dic_{16}$ by $\mathbb{Z}^2$. Note that the action of Dic_{16} on the (abelian) kernel in this extension is exactly the one on Λ from §B.2.4, which is through the quotient $Dic_{16} \twoheadrightarrow W_2$. Precisely, as in the previous proof, $a = a_{23}$ and $c = c_3$ identify with the basis of Λ from Remark B.15. However, this extension is not split, so computing its lower central series seems tricky. We can try to make it split, by considering the quotient by the relation $\sigma_1^2 = \gamma_1^2$ (which is equivalent to $a^m = 1$), but then we also kill c^{2m}, getting the finite quotient:

$$(G/A)/\sigma_1^2 \gamma_1^{-2} \cong (\Lambda/\langle ma, 2mc \rangle) \rtimes Dic_{16} \cong (\mathbb{Z}/m \times \mathbb{Z}/2m) \rtimes Dic_{16}.$$

In fact, we remark that, using the notations of Proposition B.12, we have $\langle ma, 2mc \rangle = mV \subset \Lambda$. Thus, we can deduce from Proposition B.12 a computation of $\Gamma_*^{Dic_{16}}(\mathbb{Z}/m \times \mathbb{Z}/2m) = \Gamma_*^{W_2}(\Lambda/mV)$. Namely, if $m = 2^\nu m'$, with m' odd, we have that $2^\nu V$ contains mV, and that $2^{\nu+1}V$ equals $2^\nu V$ modulo mV, so $\Gamma_*^{W_2}(\Lambda/mV)$ stops at $\Gamma_{2\nu+2}^{W_2}(\Lambda/mV) = 2^\nu V/mV$. Finally, the lower central series of the above quotient stops at Γ_k, where $k = 2v_2(m) + 2$ if m is even, and $k = 4$ is m is odd (in the latter case, note that the relative lower central series stops at the second step, but Dic_{16} is 3-nilpotent).

This gives a lower bound for the step at which the lower central series of G stops: it cannot stop before Γ_k for $k = \max\{4, 2v_2(m) + 2\}$. However, this lower bound is far from optimal: our experimental calculations using GAP [GAP21] and the package NQ [Nic96] show that, for many small values of m (all of those that we tried), the lower central series of G/A, and hence those of G and of $\mathbf{B}_{2,m}(\mathbb{P}^2)$, do not stop before Γ_{100}. We thus conjecture:

Conjecture 6.71 *Let $m \geqslant 3$ be an integer. The lower central series of $\mathbf{B}_{2,m}(\mathbb{P}^2)$ does not stop.*

A Appendix 1: Generators and relations

We recollect here the presentations of several families of groups considered in the paper which are used for our work.

Notation A.1 We write $a \rightleftarrows b$ to denote the relation saying that a and b commute.

A.1 Virtual and welded braids

A.1.1 Presentations

Proposition A.2 ([BB11, Def. 1]) *The virtual braid group* $\mathbf{vB}_n$ *admits a presentation given by a set of generators* $\{\sigma_1, \ldots, \sigma_{n-1}\}$ *satisfying the relations of the classical braid group* $\mathbf{B}_n$ *(corresponding to ordinary crossings in diagrams) and a set of generators* $\{\tau_1, \ldots, \tau_{n-1}\}$ *which satisfies the relations of the symmetric group* $\mathfrak{S}_n$ *(corresponding to virtual crossings) and two additional mixed relations:*

$$\begin{cases} (R1) \ \sigma_i \ \rightleftarrows \ \tau_k & \text{if } |i-k| \geqslant 2; \\ (R2) \ \tau_i \tau_{i+1} \sigma_i = \sigma_{i+1} \tau_i \tau_{i+1} & \text{if } i \in \{1, \ldots, n-2\}. \end{cases} \tag{A.1}$$

Proposition A.3 ([FRR97, §1][BWC07, Th. 2.1][BH13, Prop. 3.3 and 3.7]) *The welded braid group* $\mathbf{wB}_n$ *admits a presentation given by a set of generators* $\{\sigma_1, \ldots, \sigma_{n-1}\}$ *satisfying the relations of the classical braid group* $\mathbf{B}_n$ *and a set of generators* $\{\tau_1, \ldots, \tau_{n-1}\}$ *which satisfies the relations of the symmetric group* $\mathfrak{S}_n$ *and three additional mixed relations:*

$$\begin{cases} (R1) \ \sigma_i \ \rightleftarrows \ \tau_k & \text{if } |i-k| \geqslant 2; \\ (R2) \ \tau_i \tau_{i+1} \sigma_i = \sigma_{i+1} \tau_i \tau_{i+1} & \text{if } i \in \{1, \ldots, n-2\}; \\ (R3) \ \sigma_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \sigma_{i+1} & \text{if } i \in \{1, \ldots, n-2\}. \end{cases} \tag{A.2}$$

Each τ_i *corresponds to a loop in* $F_{n\mathbb{S}^1}(\mathbb{D}^3)/\mathfrak{S}_n$ *that interchanges two unknots without either of them passing through the other; each* σ_i *corresponds to a loop in* $F_{n\mathbb{S}^1}(\mathbb{D}^3)/\mathfrak{S}_n$ *that interchanges two unknots, while one passes through the other.*

The extended welded braid group $\widetilde{\mathbf{w}}\mathbf{B}_n$ *admits a presentation with generators and relations of* $\mathbf{wB}_n$ *plus additional generators* $\{\rho_1, \ldots, \rho_n\}$ *satisfying the relations of the abelian group* $(\mathbb{Z}/2\mathbb{Z})^n$ *and five extra mixed relations:*

$$\begin{cases} (R4) \ \sigma_i \ \rightleftarrows \ \rho_k & \text{if } k \notin \{i, i+1\}; \\ (R5) \ \tau_i \ \rightleftarrows \ \rho_k & \text{if } k \notin \{i, i+1\}; \\ (R6) \ \sigma_i \rho_i = \rho_{i+1} \sigma_i & \text{if } i \in \{1, \ldots, n-1\}; \\ (R7) \ \tau_i \rho_i = \rho_{i+1} \tau_i & \text{if } i \in \{1, \ldots, n-1\}; \\ (R8) \ \sigma_i \rho_{i+1} = \rho_i \tau_i \sigma_i^{-1} \tau_i & \text{if } i \in \{1, \ldots, n-1\}. \end{cases} \tag{A.3}$$

Each ρ_i *corresponds to a loop in* $F_{n\mathbb{S}^1}(\mathbb{D}^3)/W_n$ *that rotates a single circle by 180 degrees.*

A.1.2 Welded braid diagrams

There is a well-known method of visualising the presentations of §A.1.1 in terms of braid diagrams, which we recall here.

Notation A.4 We fix the convention that reading a braid word from left to right corresponds to reading a braid diagram from top to bottom. In §A.1.4, when interpreting (extended) welded braids as automorphisms of $\mathbf{F}_n$, we will read composition from the right to left (as usual for functions). Thus braid diagrams, as automorphisms of $\mathbf{F}_n$, are composed from bottom to top.

Proposition A.5 *The virtual braid group* $\mathbf{vB}_n$ *consists of braid diagrams with crossings as depicted in Figure A.1, up to planar isotopy and:*
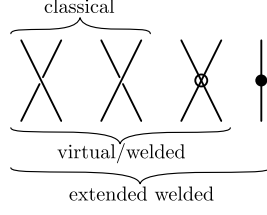
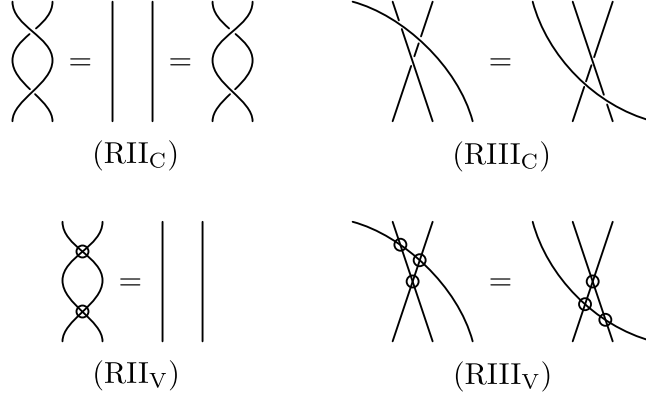Figure A.1   Crossings and decorations in braid diagrams.

Figure A.2   The classical (top row) and virtual (bottom row) Reidemeister-II and -III moves.

- *the classical and virtual Reidemeister-II and Reidemeister-III moves in Figure A.2;*
- *the virtual move* (V) *in Figure A.3.*

Note that there are obvious variants of these moves that also hold as a consequence. For example, if we write $(\overline{V})$ for the variant of (V) where the classical crossing is reversed, then $(\overline{V})$ follows from (V) and two instances of $(\mathrm{RII_C})$. Similarly, if we write $(\widehat{V})$ for the mirror image of (V), then $(\widehat{V})$ follows from $(\overline{V})$ and four instances of $(\mathrm{RII_V})$.

Proposition A.6 *The welded braid group* $\mathbf{wB}_n$ *consists of braid diagrams with crossings as depicted in Figure A.1, up to planar isotopy, the moves listed in Proposition A.5 and:*

- *the welded move* (W) *in Figure A.3.*

Proposition A.7 *The extended welded braid group* $\widetilde{\mathbf{w}}\mathbf{B}_n$ *consists of braid diagrams with crossings and marked points as depicted in Figure A.1, up to planar isotopy, the moves listed in Propositions A.5 and A.6, and:*

- *the extended welded moves in Figures A.4 and A.5.*

As above, there are variants of these moves that hold as a consequence; in particular, the mirror images of (EC), (EW), (ECW) and (ECW)′.

Sketch proof of Propositions A.5–A.7. For $\mathbf{vB}_n$, we translate the presentation (A.1) into diagrams by sending σ_i, σ_i^{-1} and τ_i respectively to the three crossings depicted in Figure A.1, adding $i-1$ trivial strands to the left and $n-i-1$ trivial strands to the right. Concatenation of elements from
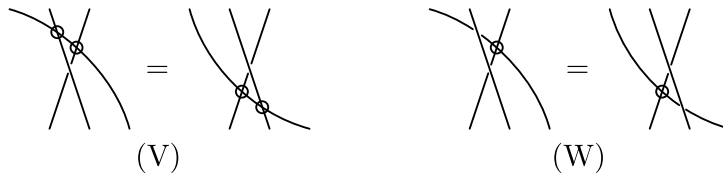

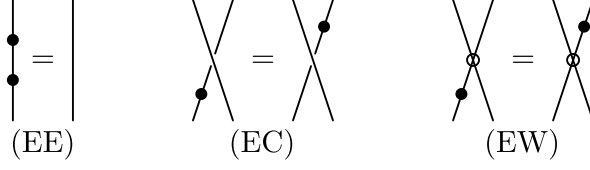Figure A.3   The virtual (V) and welded (W) moves.

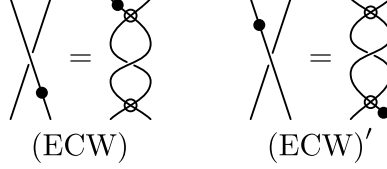Figure A.4   Three of the four extended welded moves.

Figure A.5   The fourth extended welded move (with two variants).

left to right corresponds to stacking braid diagrams from top to bottom. The classical and virtual Reidemeister-II moves correspond to the fact that σ_i^{-1} is the inverse of σ_i and that τ_i has order two. The braid relations in $\mathbf{B}_n$ and in $\mathfrak{S}_n$ correspond to the classical and virtual Reidemeister-III moves. The commutation relations in $\mathbf{B}_n$ and in $\mathfrak{S}_n$, as well as relation $(R1)$ in (A.1), correspond to planar isotopy of diagrams. Finally, relation $(R2)$ in (A.1) corresponds to the virtual move (V).

For $\mathbf{wB}_n$, we just add one further relation, $(R3)$ of (A.2), which says that $\sigma_i\sigma_{i+1}\tau_i = \tau_{i+1}\sigma_i\sigma_{i+1}$. This may be rewritten as $\sigma_{i+1}\tau_i\sigma_{i+1}^{-1} = \sigma_i^{-1}\tau_{i+1}\sigma_i$, which corresponds to the welded move (W).

For $\widetilde{\mathbf{w}}\mathbf{B}_n$, we add the marked points depicted on the right-hand side of Figure A.1, which correspond to the generators of $(\mathbb{Z}/2\mathbb{Z})^n$. The relations of $(\mathbb{Z}/2\mathbb{Z})^n$ correspond to the extended welded move (EE) (the generators have order two) and planar isotopies (the generators pairwise commute). The relations $(R4)$ and $(R5)$ also correspond to planar isotopies. The relations $(R6)$–$(R8)$ correspond to the extended welded moves (EC), (EW) and (ECW). The variant $(ECW)'$ of (ECW) is actually a consequence, following from (ECW) along with two instances each of (EW), (RII_C) and (RII_V). $\quad\square$

A.1.3   Pure virtual and welded braids

A presentation of the group of pure welded braids has been known since the work of McCool in the 1980's, which uses geometric group theory. Another proof was given by Fenn, Rimányi and Rourke towards the end of the century [FRR97]:

Proposition A.8 ([McC86; Gol81]) *The pure welded braid group* $\mathbf{wP}_n$ *admits a presentation given by generators* $\{\chi_{ij} \mid 1 \leqslant i, j \leqslant n, i \neq j\}$ *subject to the following relations, for* i, j, k, l *pairwise distinct:*

$$\begin{cases} (PR1)\ \chi_{ij} \ \rightleftarrows \ \chi_{kl}; \\ (PR2)\ \chi_{ik} \ \rightleftarrows \ \chi_{jk}; \\ (PR3)\ \chi_{ij} \ \rightleftarrows \ \chi_{ik}\chi_{jk}. \end{cases} \tag{A.4}$$

More recently, using the Reidemeister-Schreier method, Bardakov gave a presentation of the pure virtual braid group, which looks like the previous one, with one relation removed (χ_{ik} and χ_{jk} no longer commute):

Proposition A.9 ([Bar04]) *The pure virtual braid group* $\mathbf{vP}_n$ *admits a presentation given by generators* $\{\chi_{ij} \mid 1 \leqslant i, j \leqslant n, i \neq j\}$ *subject to the following relations, for* i, j, k, l *pairwise distinct:*

$$\begin{cases} (PR1')\ \chi_{ij} \ \rightleftarrows \ \chi_{kl}; \\ (PR2')\ \chi_{ij}\chi_{ik}\chi_{jk} = \chi_{jk}\chi_{ik}\chi_{ij}. \end{cases} \tag{A.5}$$

Corollary A.10 *The projection* $\mathbf{vP}_n \twoheadrightarrow \mathbf{wP}_n$ *induces an isomorphism* $\mathbf{vP}_n^{\mathrm{ab}} \cong \mathbf{wP}_n^{\mathrm{ab}}$; *both are free on the classes of the* χ_{ij} *for* $1 \leqslant i \neq j \leqslant n$.

Remark A.11 The corollary can also be proved directly: showing that the χ_{ij} generate $\mathbf{vP}_n$ (hence also $\mathbf{wP}_n$) is not difficult. Then, the linear independence of the classes of the χ_{ij} in $\mathbf{wP}_n^{\mathrm{ab}}$ (hence also in $\mathbf{vP}_n^{\mathrm{ab}}$) readily follows from the following fact: for any choice of $k \neq l$, there is a projection $\pi_{k,l} : \mathbf{wP}_n^{\mathrm{ab}} \twoheadrightarrow \mathbf{wP}_2^{\mathrm{ab}} \cong \mathbb{Z}^2$ (using $\mathbf{wP}_2 \cong \mathrm{Inn}(\mathbf{F}_2) \cong \mathbf{F}_2$) killing all the χ_{ij}, save χ_{kl} and χ_{lk}, which are sent to a basis of $\mathbb{Z}^2$. Such a projection is induced by the map $\mathbf{wP}_n \twoheadrightarrow \mathbf{wP}_2$ obtained by forgetting all the strands except the j-th and the k-th ones.

A.1.4 Welded braids as automorphisms of the free group

Let us use the following notations for automorphisms of the free group $\mathbf{F}_n$ on n generators $x_1, ..., x_n$:

$$\tau_i : \begin{cases} x_i & \longmapsto & x_{i+1} \\ x_{i+1} & \longmapsto & x_i \end{cases} \quad , \quad \sigma_i : \begin{cases} x_i & \longmapsto & x_i x_{i+1} x_i^{-1} \\ x_{i+1} & \longmapsto & x_i \end{cases} \quad \text{and} \quad \rho_i : x_i \mapsto x_i^{-1},$$

where it is implicit that the x_k that are not mentioned are fixed. The groups $\mathbf{wB}_n$ and $\widetilde{\mathbf{w}}\mathbf{B}_n$ may be seen as subgroups of $\mathrm{Aut}(\mathbf{F}_n)$ as follows; we refer the reader for further details to [Gol81, Th. 5.3], or to [Dam17, Cor. 4.3], or else to [FRR97, §1] and [BWC07, Th. 2.1].

Theorem A.12 *The morphism $\mathbf{wB}_n \hookrightarrow \widetilde{\mathbf{w}}\mathbf{B}_n \to \mathrm{Aut}(\mathbf{F}_n)$ sending τ_i, σ_i and ρ_i to the corresponding automorphisms (as defined above) is a well-defined isomorphism onto its image, which consists of all automorphisms of the form:*

$$x_i \longmapsto w_i x_{\sigma(i)}^{\pm 1} w_i^{-1}, \quad \text{with } w_1, \dots, w_n \in \mathbf{F}_n \text{ and } \sigma \in \mathfrak{S}_n.$$

In particular, the generators χ_{ij} of the pure welded braid group $\mathbf{wP}_n$ are identified with:

$$\chi_{ij} : x_k \mapsto \begin{cases} x_j x_i x_j^{-1} & \text{if } k = i \\ x_k & \text{else.} \end{cases}$$

In particular, we have $\chi_{i,i+1} = \tau_i \sigma_i$ (where, as usual for functions, composition of automorphisms is written from right to left).

Remark A.13 Our definition of σ_i disagrees with the one in [Dam17]; instead it corresponds to the *inverse* of the element called σ_i in [Dam17]. This is because our $\mathrm{Aut}(\mathbf{F}_n)$ is the opposite group of the group called $\mathrm{Aut}(\mathbf{F}_n)$ in [Dam17], since we view composition of automorphisms of $\mathbf{F}_n$ from the right to left, whereas composition of automorphisms goes from left to right in [Dam17]; see Remark 6.3 of [Dam17]. On the other hand, these considerations about the direction of composition do not affect the elements τ_i and ρ_i, since they are self-inverse. (We warn the reader, however, that our τ_i and ρ_i correspond, respectively, to the elements denoted by ρ_i and τ_i in [Dam17].)

A.2 Braids on surfaces

The main tool for determining presentations of surface braid groups (including braids on the disc, which are usual Artin braids) are the Fadell-Neuwirth exact sequences; see Proposition 6.25. These were already used in the course of the proof of Proposition 6.8, where they were used to obtain generators of these groups. Let us now briefly explain how they may be used in order to determine defining relations on these generators. For S a non-closed surface, these exact sequences give a decomposition of $\mathbf{B}_{1,n-1}(S)$ as a semi-direct product of $\mathbf{B}_{n-1}(S)$ with a free group F. Then suppose that one has a set of relations satisfied in $\mathbf{B}_n(S)$, defining a group G_n and a well-defined surjection $\pi : G_n \twoheadrightarrow \mathbf{B}_n(S)$. One can consider the subgroup $G_{1,n-1} = \pi^{-1}(\mathbf{B}_{1,n-1}(S))$, use the relations to show that it is of index at most n in G_n (which implies that the induced surjection of $G_n / G_{1,n-1}$ onto $\mathbf{B}_n(S)/\mathbf{B}_{1,n-1}(S)$ is a bijection), and determine a presentation of $G_{1,n-1}$ by using the Reidemeister-Schreier method. Then one shows that $G_{1,n-1}$ decomposes as a semi-direct product of a quotient isomorphic to G_{n-1} with a kernel K generated by a family of elements sent by π to a basis of the free group F. The latter fact implies that this family must be a free basis of K, which means that $\pi : K \to F$ is an isomorphism. By induction, $\pi : G_{n-1} \to \mathbf{B}_{n-1}(S)$ is an

isomorphism. Then $\pi\colon G_{1,n-1} \to \mathbf{B}_{1,n-1}(S)$ must be an isomorphism too. And since the induced surjection of $G_n/G_{1,n-1}$ onto $\mathbf{B}_n(S)/\mathbf{B}_{1,n-1}(S)$ is a bijection, $\pi\colon G_n \to \mathbf{B}_n(S)$ is an isomorphism.

This method can be used to get presentations of the braid groups of every non-closed surface of finite type; see [Bel04] for instance. It can also be adapted to the case of closed surfaces, replacing semi-direct product decompositions by non-split extensions, with a care for the exceptional cases where this is not even an extension. However, we give here a direct general argument presenting $\mathbf{B}_n(S)$ as the quotient of $\mathbf{B}_n(S-\mathrm{pt})$ by one explicit relation, by applying the Seifert-Van-Kampen theorem to configuration spaces; see Proposition A.17.

A.2.1   Surfaces with one boundary component

A presentation of braid groups of compact surfaces with one boundary component can be found in [HL02, §4] and in [Bel04, Th. 1.1 and A.2]. We re-write them with our own conventions, which we now explain.

Let us denote by $\Sigma_{g,1}$ the orientable connected compact surface of genus g with one boundary component, and by $\mathcal{N}_{g,1}$ the non-orientable connected compact surface of genus g with one boundary component. We draw $\Sigma_{g,1}$ as a rectangle with $2g$ handles attached to it, and $\mathcal{N}_{g,1}$ as a rectangle with g crosscaps. Our notations for braid generators are detailed in Figure A.6. Our drawings of braids are to be thought of as seen from above, and the left-to-right direction in products corresponds to the foreground-to-background direction in our drawings. For instance, with the notations of Figure A.6, we have that $\sigma_1 a_k^{(i)} \sigma_1^{-1} = a_k^{(i+1)}$. As an illustration of these conventions, we draw different representations of the pure braid generators in Figure A.7.
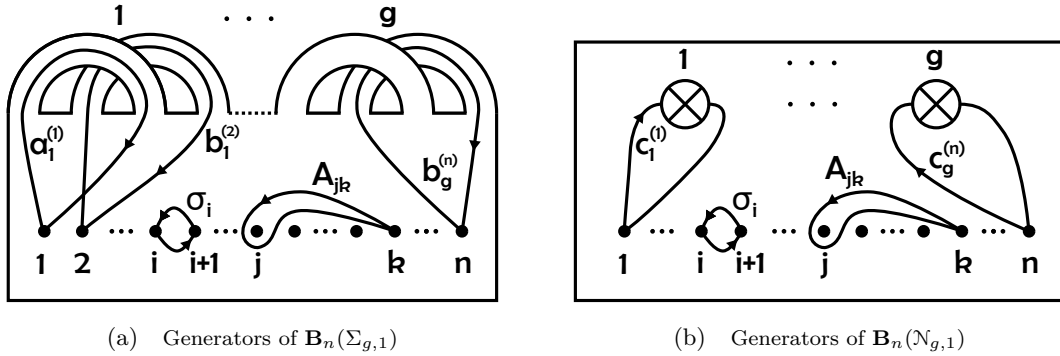

(a)   Generators of $\mathbf{B}_n(\Sigma_{g,1})$ (b)   Generators of $\mathbf{B}_n(\mathcal{N}_{g,1})$

Figure A.6   **Generators of surface braid groups.** For each generator (except for σ_i, where the points i and $i+1$ move), only one point of the configuration moves, and the others stay put. We often denote $x_k^{(1)}$ by x_k, for $x = a, b, c$.
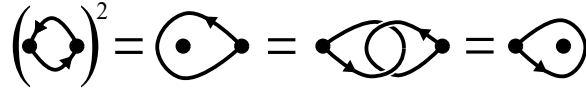

Figure A.7   The standard pure braid generator $A_{i,i+1} = \sigma_i^2$.

Proposition A.14 *Let $g \geqslant 0$. A presentation of the braid group on $\Sigma_{g,1}$, generated by $\sigma_1, ..., \sigma_{n-1}$, $a_1, ..., a_g$, $b_1, ..., b_g$, is given by the braid relations for $\sigma_1, ..., \sigma_{n-1}$, to which are added the following four families of relations (where x and y denote either a or b, and $1 \leqslant r, s \leqslant g$):*

$$\begin{cases} (BS1) & \sigma_i \;\rightleftarrows\; x_r & \textit{for all } r \textit{ and all } i \geqslant 2 \text{ ;} \\ (BS2) & x_r \;\rightleftarrows\; \sigma_1 y_s \sigma_1^{-1} & \textit{for } r < s \text{ ;} \\ (BS3) & (\sigma_1 x_r)^2 = (x_r \sigma_1)^2 & \textit{for all } r \text{ ;} \\ (BS4) & [\sigma_1 b_r \sigma_1^{-1}, a_r^{-1}] = \sigma_1^2 & \textit{for all } r. \end{cases} \tag{A.6}$$

It is easy to check that these relations hold in $\mathbf{B}_n(\Sigma_{g,1})$ by drawing explicit isotopies. See for instance Figure 6.2 for a drawing of $(BS4)$ (which generalises to $[b_s^{(j)}, (a_r^{(j)})^{-1}] = A_{rs}$ if $r < s$).

Proposition A.15 *Let $g \geqslant 0$. A presentation of the braid group on $\mathbb{N}_{g,1}$, generated by $\sigma_1, ..., \sigma_{n-1}$, $c_1, ..., c_g$ is given by the braid relations for $\sigma_1, ..., \sigma_{n-1}$, to which are added the following three families of relations (where $1 \leqslant r, s \leqslant g$):*

$$\begin{cases} (BN1) & \sigma_i \rightleftarrows c_r & \text{for all } r \text{ and all } i \geqslant 2 \text{ ;} \\ (BN2) & c_r \rightleftarrows \sigma_1 c_s \sigma_1^{-1} & \text{for } r < s \text{ ;} \\ (BN3) & [\sigma_1 c_r \sigma_1^{-1}, c_r^{-1}] = \sigma_1^2 & \text{for all } r. \end{cases} \tag{A.7}$$

Here again, it is easy to check these relations explicitly. See for instance Figure 6.1 for a drawing of $(BN3)$ (which generalises to $[c_s^{(j)}, (c_r^{(j)})^{-1}] = A_{rs}$ if $r < s$).

Let us denote by $\mathbb{N}_{g,n+1}$ the non-orientable connected compact surface of genus g with either $n + 1$ punctures or $n + 1$ boundary components (recall that, up to isotopy, removing a point and removing an open disc are equivalent). The braid group $\mathbf{B}_m(\mathbb{N}_{g,n+1})$ may be seen as a subgroup of $\mathbf{B}_{m+n}(\mathbb{N}_{g,1})$; see Proposition 6.25. Namely, this subgroup is generated by $\sigma_1, ..., \sigma_{n-1}, c_1, ..., c_g$ and $z_j := A_{1,m+j}$ for all $1 \leqslant j \leqslant n$. We need the following result in the proof of Proposition 6.36:

Proposition A.16 *Let $g \geqslant 0$ and $m \geqslant 1$. A presentation of the braid group $\mathbf{B}_m(\mathbb{N}_{g,n+1})$, generated by $\sigma_1, ..., \sigma_{m-1}, c_1, ..., c_g, z_1, ..., z_n$ is given by the relations from Proposition A.15, together with the following four families of relations:*

$$\begin{cases} (BN4) & z_j \rightleftarrows \sigma_i & \text{for all } j \leqslant n \text{ and all } i \in \{2, ..., m-1\} \text{ ;} \\ (BN5) & c_r \rightleftarrows \sigma_1 z_j \sigma_1^{-1} & \text{for all } j \leqslant n \text{ and all } r \leqslant g \text{ ;} \\ (BN6) & z_i \rightleftarrows \sigma_1 z_j \sigma_1^{-1} & \text{for } i > j \text{ ;} \\ (BN7) & (\sigma_1 z_j)^2 = (z_j \sigma_1)^2 & \text{for all } j \leqslant n. \end{cases}$$

Once more, these relations are easy to check explicitly.

A.2.2 Closed surfaces

When S is a closed surface, one needs to add a single relation to a presentation of $\mathbf{B}_n(S - \text{pt})$ to get a presentation of $\mathbf{B}_n(S)$. In fact, this is a very general fact, which does not require any hypothesis on the surface.

Proposition A.17 *Let S be a connected surface and $x \in S$ any point in its interior. The inclusion of $S - x$ into S induces a surjective homomorphism*

$$\mathbf{B}_n(S - x) \twoheadrightarrow \mathbf{B}_n(S)$$

whose kernel is normally generated by a single element β. Explicitly, β is a braid with $n - 1$ trivial strands, whose remaining strand loops once around the puncture x.

Proof. Choose a subdisc $D \subset S$ containing x in its interior, and a metric on D. Write $\mathcal{U}_n(S, x)$ for the subspace of $C_n(S)$ of configurations that have a unique closest point in D to x (which may be x itself). Together with $C_n(S - x)$, this forms an open cover

$$\{\mathcal{U}_n(S, x), C_n(S - x)\}$$

of $C_n(S)$, with intersection $\mathcal{U}_n(S, x) \cap C_n(S - x) = \mathcal{U}_n(S - x, x)$ the space of n-point configurations in $S - x$ that have a unique closest point in $D - x$ to x. Note that these subspaces of $C_n(S)$ are all path-connected. Let us choose a basepoint for $C_n(S)$ that lies in $\mathcal{U}_n(S - x, x)$. The Seifert-van Kampen theorem then gives us a pushout square of groups:

$$\begin{array}{ccc} \pi_1(\mathcal{U}_n(S - x, x)) & \longrightarrow & \mathbf{B}_n(S - x) \\ \downarrow & & \downarrow \\ \pi_1(\mathcal{U}_n(S, x)) & \longrightarrow & \mathbf{B}_n(S). \end{array} \tag{A.8}$$

There is a well-defined projection $\mathcal{U}_n(S,x) \twoheadrightarrow D$ given by remembering just the unique closest point in D to x, which restricts to a projection $\mathcal{U}_n(S-x,x) \twoheadrightarrow D-x$. These are both locally trivial fibrations with fibres canonically homeomorphic to $C_{n-1}(S-x)$. (Over a point $p \in D$, the homeomorphism is induced by the evident homeomorphism between $S-x$ and $S - \bar{B}_{d(p,x)}(x)$, where $\bar{B}_r(x)$ denotes the closed ball of radius r in D centered at x and $d(-,-)$ is the metric that we chose on D.) The inclusion of the latter into the former is therefore a map of locally trivial fibrations which is the identity on fibres. Considering the induced long exact sequences of homotopy groups, we obtain a map of exact sequences:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{B}_{n-1}(S-x) & \longrightarrow & \pi_1(\mathcal{U}_n(S-x,x)) & \longrightarrow & \pi_1(D-x) \cong \mathbb{Z} & \longrightarrow & 1 \\
& & \cong \downarrow \quad \nwarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{B}_{n-1}(S-x) & \underset{\cong}{\xrightarrow{\quad r \quad}} & \pi_1(\mathcal{U}_n(S,x)) & \longrightarrow & \pi_1(D) = 1. & &
\end{array}
\tag{A.9}
$$

The map r obtained from this diagram is a retraction for the upper short exact sequence, whose existence implies that $\pi_1(\mathcal{U}_n(S-x,x))$ is the direct product of $\pi_1(\mathcal{U}_n(S,x))$ and $\mathbb{Z}$. Then, the left-hand vertical map in (A.8) identifies with the projection that forgets the $\mathbb{Z}$ factor. Together with the fact that (A.8) is a pushout square, this implies that the right-hand vertical map in (A.8) is the quotient of $\mathbf{B}_n(S-x)$ by (the subgroup normally generated by) the image of the $\mathbb{Z}$ factor of $\pi_1(\mathcal{U}_n(S-x,x))$ in $\mathbf{B}_n(S-x)$. We may choose for a generator of this $\mathbb{Z}$ factor any element of $\pi_1(\mathcal{U}_n(S-x,x))$ that projects to a generator of $\pi_1(D-x)$, for example the braid described in the proposition. $\qquad\square$

Let us make this explicit:

Corollary A.18 *For all $g \geqslant 0$, the braid group $\mathbf{B}_n(\Sigma_g)$ is the quotient of $\mathbf{B}_n(\Sigma_{g,1})$ by the relation:*

$$
\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1 \;=\; \prod_{r=1}^{g}[a_r, b_r^{-1}].
$$

Similarly, the braid group $\mathbf{B}_n(\mathcal{N}_g)$ is the quotient of $\mathbf{B}_n(\mathcal{N}_{g,1})$ by the relation:

$$
\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1 \;=\; c_1^2 \cdots c_g^2.
$$

We note that we recover as a particular case the usual presentations of the braid group on the sphere (see [FV62] or [Bir74, Th. 1.11]) and of the projective plane (see [Bus66, §III, page 83]). Not having to treat these as exceptional cases is one of the great advantages of the present method.

Proof of Corollary A.18. It is a direct application of Proposition A.17, using the fact that $\Sigma_g - \mathrm{pt}$ (resp. $\mathcal{N}_g - \mathrm{pt}$) is isotopy equivalent to $\Sigma_{g,1}$ (resp. to $\mathcal{N}_{g,1}$). We note that $\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1 = A_{12}\cdots A_{1n}$ is the (pure) braid obtained by making the first strand turn once around all the other ones; see Figure A.8 for the relevant drawings. $\qquad\square$

A.2.3 Partitioned braids on closed surfaces

The above proof of Proposition A.17 generalises to partitioned braid groups without much difficulty; see Remark A.23. However, we prefer to deduce these generalisations directly from Proposition A.17 itself. We begin with the case of pure braids by describing a direct equivalence between Proposition A.17 and the following statement:

Proposition A.19 *Let S be a connected surface and $x \in S$ any point in its interior. The inclusion of $S-x$ into S induces a surjective homomorphism*

$$
\mathbf{P}_n(S-x) \twoheadrightarrow \mathbf{P}_n(S)
$$

whose kernel is normally generated by n elements $\beta_1, ..., \beta_n$. Explicitly, β_i is a braid whose i-th strand loops once around the puncture x, the other strands being trivial.

(a) $\quad \partial = [a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}] A_{1n}^{-1} \cdots A_{12}^{-1}$

(b) $\quad \partial = c_1^2 \cdots c_g^2 A_{1n}^{-1} \cdots A_{12}^{-1}$
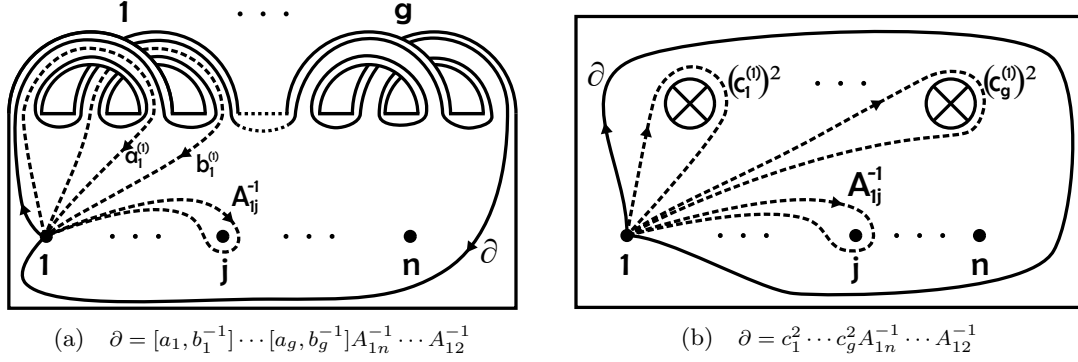
Figure A.8   The boundary elements in $\mathbf{B}_n(\Sigma_{g,1})$ and $\mathbf{B}_n(\mathcal{N}_{g,1})$.

Equivalence between Proposition A.17 and Proposition A.19. Let N be the subgroup of $\mathbf{B}_n(S-x)$ normally generated by the braid β (from Proposition A.17). Note that β is a pure braid, hence we have $N \subseteq \mathbf{P}_n(S-x)$. Now, let N' be the subgroup of $\mathbf{P}_n(S-x)$ normally generated by the β_i's. We will show that $N = N'$. This implies the equivalence of Propositions A.17 and A.19 by considering the diagram

$$
\begin{array}{ccccccc}
N' & \lhook\joinrel\longrightarrow & \ker(\pi_{\mathbf{P}}) & \lhook\joinrel\longrightarrow & \mathbf{P}_n(S-x) & \xrightarrow{\ \pi_{\mathbf{P}}\ } & \mathbf{P}_n(S) \\
& & \Big\| & & \Big\downarrow & & \Big\downarrow \\
N & \lhook\joinrel\longrightarrow & \ker(\pi_{\mathbf{B}}) & \lhook\joinrel\longrightarrow & \mathbf{B}_n(S-x) & \xrightarrow{\ \pi_{\mathbf{B}}\ } & \mathbf{B}_n(S) \\
& & & & \Big\downarrow & & \Big\downarrow \\
& & & & \mathfrak{S}_n & =\!=\!=\!=\!= & \mathfrak{S}_n
\end{array}
$$

and noting that the two propositions are equivalent, respectively, to the statements $N = \ker(\pi_{\mathbf{B}})$ and $N' = \ker(\pi_{\mathbf{P}})$.

Our definitions of β and the β_i are up to some choices, but all these choices give elements conjugate to each other (in $\mathbf{B}_n(S-x)$ or in $\mathbf{P}_n(S-x)$, respectively) or each other's inverses, whic@h does not affect the definition of N and N'. We can make these choices so that:

- the only moving strand of β is the first one,
- β commutes with every element of the subgroup $\mathbf{B}_{1,n-1} = \langle \sigma_2, ..., \sigma_{n-1} \rangle$ of $\mathbf{B}_n(S-x)$ (which consists of braids in a fixed disc $\mathbb{D} \subset S - x$ involving only the strands 2 to n),
- for each i, $\beta_i = (\sigma_1 \cdots \sigma_{i-1})^{-1} \beta (\sigma_1 \cdots \sigma_{i-1})$.

See Figure A.9 for an example of such choices. The latter relations imply $N' \subseteq N$.

We now show that N' contains all the conjugates of β by elements of $\mathbf{B}_n(S-x)$, which implies $N' \supseteq N$. In order to do this, we need only show that it contains $t^{-1}\beta t$ for t in a set of representatives of classes modulo $\mathbf{P}_n(S-x)$: then every element of $\mathbf{B}_n(S-x)$ is of the form $t\alpha$ for some such t and some $\alpha \in \mathbf{P}_n(S-x)$, and $(t\alpha)^{-1}\beta(t\alpha) = \alpha^{-1}(t^{-1}\beta t)\alpha$ must be in N'.

Every element $\tau \in \mathfrak{S}_n \cong \mathbf{B}_n(S-x)/\mathbf{P}_n(S-x)$ is the product of an element τ' fixing 1 with some cycle $\tau_1 \cdots \tau_{i-1}$ (precisely, $i = \tau(1)$). Since β commutes with every element of the subgroup $\langle \sigma_2, ..., \sigma_{n-1} \rangle \subset \mathbf{B}_n(S-x)$, and since this subgroup surjects onto permutations fixing 1, we can choose a lift t' of τ' commuting with β, so that the lift $t = t'\sigma_1 \cdots \sigma_{i-1}$ of τ to $\mathbf{B}_n(S-x)$ satisfies:

$$ t^{-1}\beta t = (\sigma_1 \cdots \sigma_{i-1})^{-1} t'^{-1} \beta t' (\sigma_1 \cdots \sigma_{i-1}) = (\sigma_1 \cdots \sigma_{i-1})^{-1} \beta (\sigma_1 \cdots \sigma_{i-1}) = \beta_i \in N', $$

whence our result. $\qquad\square$

It is not difficult to generalise this to any partitioned braid group:

Corollary A.20 *Let S be a connected surface, $x \in S$ any point in its interior and $\lambda = (n_1, \ldots, n_l)$ be a partition of n of length l. The inclusion of $S - x$ into S induces a surjective homomorphism*

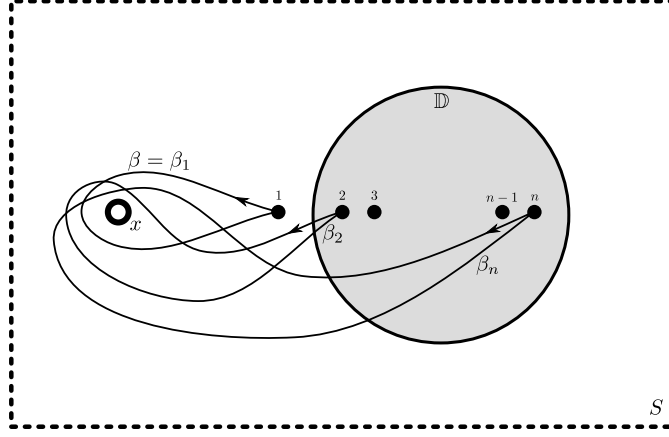$$ \mathbf{B}_\lambda(S-x) \twoheadrightarrow \mathbf{B}_\lambda(S) $$

Figure A.9   Braids β and $\beta_1, \ldots, \beta_n$ from the proof of the equivalence of Propositions A.17 and A.19.

whose kernel is normally generated by l elements $\beta_1, \ldots, \beta_l$. Explicitly, β_i is a braid with $n-1$ trivial strands, except one block in the i-th block which loops once around the puncture x.

Proof. We have $\mathbf{P}_n(S-x) \subset \mathbf{B}_\lambda(S-x) \subset \mathbf{B}_n(S-x)$. As a consequence, if N is a normal subgroup of $\mathbf{P}_n(S-x)$ which is normal in $\mathbf{B}_n(S-x)$, we have $\mathbf{P}_n(S-x)/N \subset \mathbf{B}_\lambda(S-x)/N \subset \mathbf{B}_n(S-x)/N$. Moreover, normal generators for N in $\mathbf{P}_n(S-x)$ are also normal generators for N in $\mathbf{B}_\lambda(S-x)$ and, in fact, fewer generators are needed to generate normally N in $\mathbf{B}_\lambda(S-x)$. Namely, using the notations from the previous proof, we have $\beta_{\alpha+1} = \sigma_\alpha^{-1} \beta_\alpha \sigma_\alpha$, so that $\beta_{\alpha+1}$ and β_α are conjugate in $\mathbf{B}_\lambda(S-x)$ whenever $\sigma_\alpha \in \mathbf{B}_\lambda(S-x)$, which happens when α and $\alpha+1$ are in the same block of λ. As a consequence, we need to pick only one index α in each block of λ in order for the β_α to normally generate N in $\mathbf{B}_\lambda(S-x)$. $\qquad\square$

Remark A.21 This boils down to considering representatives modulo $\mathbf{B}_\lambda(S-x)$ instead of modulo $\mathbf{P}_n(S-x)$ (that is, elements of $\mathfrak{S}_n/\mathfrak{S}_\lambda$ instead of $\mathfrak{S}_n$) in the previous reasoning. In fact, one can see that there are straightforward equivalences between all these statements for the different partitions of n.

Let us make explicit these statements (using the usual convention $A_{ji} = A_{ij}$ and $A_{ii} = 1$):

Corollary A.22 *For all $g \geqslant 0$ and any partition $\lambda = (n_1, ..., n_l)$, the braid group $\mathbf{B}_\lambda(\Sigma_g)$ is the quotient of $\mathbf{B}_\lambda(\Sigma_{g,1})$ by the relations:*

$$A_{\alpha 1} \cdots A_{\alpha n} \;=\; \prod_{r=1}^{g} [a_r^{(\alpha)}, (b_r^{(\alpha)})^{-1}].$$

Similarly, the braid group $\mathbf{B}_\lambda(\mathcal{N}_g)$ is the quotient of $\mathbf{B}_\lambda(\mathcal{N}_{g,1})$ by the relations:

$$A_{\alpha 1} \cdots A_{\alpha n} \;=\; (c_1^{(\alpha)})^2 \cdots (c_g^{(\alpha)})^2.$$

In both cases, α runs through any set of representatives of the blocks of λ.

Remark A.23 Instead of the reasoning above, one could adapt the proof of Proposition A.17 to partitioned configuration spaces. Precisely, we get an open cover $\{\mathcal{U}_\lambda(S,x), C_\lambda(S-x)\}$ of $C_\lambda(S) = F_n(S)/\mathfrak{S}_\lambda$ by an obvious adaptation of the definition of $\mathcal{U}_n(S,x)$. However, this time $\mathcal{U}_\lambda(S,x)$ is disconnected, with one path-component for each block of the partition λ. As a consequence, one needs to apply the Seifert-van-Kampen theorem once for each path-component of $\mathcal{U}_\lambda(S,x)$, resulting in taking the quotient of the fundamental group by one additional relation for each application of the theorem. In doing so, one needs to be careful about basepoints, since one obviously cannot choose a common basepoint in the different path-components of $\mathcal{U}_\lambda(S,x)$.

A.3 Presentation of an extension

Let G be a group, which is an extension of a quotient K by a normal subgroup H. Suppose that presentations of H and K are known, namely $H = \langle X|R \rangle$ and $K = \langle Y|S \rangle$, where R is a subset of the free group $F[X]$ (resp. $S \subset F[Y]$). For each y in Y, let us fix a lift $\tilde{y}$ of the corresponding generator of K to an element of G. Then a presentation of G is given by

$$G = \langle X \sqcup Y | R \cup \widetilde{S} \cup T \rangle,$$

where $\widetilde{S}$ and T are obtained as follows:

- Each $s \in S$ is a word in the elements of Y and their inverses. If we replace each y in s by its chosen lift $\tilde{y}$, we get an element $\tilde{s}$ of G, which is in fact in H, since its projection to K is trivial by construction. Each element of H is represented by a word on the elements of X and their inverses, so we can choose some $w_s \in F[X]$ representing $\tilde{s}$. Then $\widetilde{S}$ is the following set of relations:
$$\widetilde{S} := \{ s w_s^{-1} \mid s \in S \} \subset F[X \sqcup Y].$$

- For each $y \in Y$ and each $x \in X$, the element $\tilde{y} x \tilde{y}^{-1}$ is an element of H, which can be represented by a word $w_{x,y} \in F[X]$. Then we define:
$$T := \{ y x y^{-1} w_{x,y}^{-1} \mid x \in X, \; y \in Y \} \subset F[X \sqcup Y].$$

Remark A.24 If the presentations of H and K are finite, this construction gives a finite presentation of G.

Remark A.25 (Split extensions) When the extension splits (that is, when G is a semi-direct product of K by H), one usually chooses the lifts of generators of K to be their images under a fixed section. Then the presentation obtained is somewhat simpler, since the relations in S hold in G (that is, $\widetilde{S} = S$).

Proposition A.26 *The above presentation is indeed a presentation of the extension G.*

Proof. Let G_0 be the group defined by the above presentation. By construction, the assignments $x \mapsto x \in H \subset G$ and $y \mapsto \tilde{y}$ induce a well-defined morphism π from G_0 to G. Let H_0 be the subgroup of G_0 generated by X. The morphism π restricts to a morphism $\pi_H \colon H_0 \to H$. Since the relations R are satisfied in H_0, we can construct an inverse to π_H: it is an isomorphism.

The relations T ensure that H_0 is stable by left conjugation by the $y \in G_0$. Moreover, for all $h \in H$, $\pi_H(yhy^{-1}) = \pi(y)\pi(h)\pi(y)^{-1} = \tilde{y}\pi_H(h)\tilde{y}^{-1}$. Since H is normal in G, left conjugation by $\tilde{y}$ is an automorphism of H. Since π_H is an isomorphism, left conjugation by $y \in G_0$ must be an automorphism of H_0, which implies that H_0 is stable by left conjugation by y^{-1}. Finally, H_0 is stable by left conjugation by $y^{\pm 1}$, and also (clearly) by $x^{\pm 1}$, so it is normal in G_0.

As a consequence, π induces a morphism of extensions:

$$
\begin{array}{ccccc}
H_0 & \lhook\joinrel\longrightarrow & G_0 & \longrightarrow\!\!\!\!\!\rightarrow & G_0/H_0 \\
\downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} \\
H & \lhook\joinrel\longrightarrow & G & \longrightarrow\!\!\!\!\!\rightarrow & K.
\end{array}
$$

The relations R and T become trivial in G_0/H_0, and $\widetilde{S}$ reduces to S there, so this quotient admits the presentation $\langle Y|S \rangle$. This implies that $\overline{\pi}$ is an isomorphism. Since we already know that π_H is an isomorphism, the Five Lemma allows us to conclude that π is an isomorphism. $\square$

Remark A.27 We can replace some of the generators y by their inverses before doing this construction, so we can choose T to encode either left conjugation by y, or right conjugation by y, for each y.

Corollary A.28 *If G is a 2-nilpotent group whose abelianisation is free abelian, then G is determined (up to isomorphism) by its associated Lie ring.*

Proof. We construct a presentation of G which depends only on the structure of $\mathcal{L}(G)$ (and on some choices not involving elements of G). The group G is an extension of $G^{\mathrm{ab}} = \mathcal{L}_1(G)$ by $\Gamma_2(G) = \mathcal{L}_2(G)$, to which we can apply the previous construction. Since G^{ab} is free abelian on some set Y, a presentation of this group is given by generators Y and relations $\{[y, z] \mid y, z \in Y\}$. Let $\langle X|R \rangle$ be a presentation of the group $\mathcal{L}_2(G)$. Then G admits the presentation with generators $X \sqcup Y$ and relations $R \cup \widetilde{S} \cup T$, constructed as above. We need to show that S and T can be recovered from calculations in $\mathcal{L}(G)$ alone.

- Let $s \in S$, that is, $s = [y, z]$ for some $y, z \in Y$. Then, by definition of $\mathcal{L}(G)$, $\tilde{s} = [\tilde{y}, \tilde{z}]$ is the element of $\Gamma_2(G) = \mathcal{L}_2(G)$ given by the bracket of $y, z \in \mathcal{L}_1(G)$.
- Since $[G, \Gamma_2(G)] = \{1\}$, the set T consists of the relations $[y, x]$, for $x \in R$ and $y \in S$.

Thus the above presentation of G can be obtained from the data of $\mathcal{L}(G)$, as claimed. $\square$

Remark A.29 Corollary A.28 is not true in general if the abelianisation of G is not free. For example, the dihedral group D_8 of order 8 and the quaternion group Q_8 are not isomorphic, but they are 2-nilpotent groups whose Lie rings are isomorphic. Indeed, in both cases, we have $\mathcal{L}_1 G = (\mathbb{Z}/2)^2$ and $\mathcal{L}_2 G = \mathbb{Z}/2$ and the Lie structure is fully determined by saying that whenever a and b are two distinct non-trivial elements of $\mathcal{L}_1 G$, then $[a, b]$ is non-trivial in $\mathcal{L}_2 G$.

B   Appendix 2: Some calculations of lower central series

B.1   Relative lower central series

In order to get actual computations, we need to recall some material from [Dar21, §3] about the lower central series of a semi-direct product.

Definition B.1 Let G be a group, of which H is a normal subgroup. We define the *relative lower central series* $\Gamma_*^G(H)$ by:
$$\begin{cases} \Gamma_1^G(H) := H, \\ \Gamma_{k+1}^G(H) := [G, \Gamma_k^G(H)]. \end{cases}$$

If G is the semi-direct product of H with a group K, we write $\Gamma_*^K(H)$ for $\Gamma_*^{H \rtimes K}(H)$ (which does not cause any confusion: if H is a normal subgroup of a group G, then $\Gamma_*^G(H) = \Gamma_*^{H \rtimes G}(H)$, for the semi-direct product associated to the conjugation action of G on H). It was shown in [Dar21] that in this case:
$$\Gamma_*(H \rtimes K) = \Gamma_*^K(H) \rtimes \Gamma_*(K).$$

Moreover, the filtration $\Gamma_*^K(H) = H \cap \Gamma_*(H \rtimes K)$ does have the property that $[\Gamma_i^K(H), \Gamma_j^K(H)] \subseteq \Gamma_{i+j}^K(H)$ (for all $i, j \geqslant 1$), which allows one to define an associated graded Lie ring $\mathcal{L}^K(H)$ (with brackets induced by commutators, as in §1.2). Then, the Lie ring of $H \rtimes K$ decomposes into a semi-direct product of Lie rings:
$$\mathcal{L}(H \rtimes K) = \mathcal{L}^K(H) \rtimes \mathcal{L}(K).$$

This is in fact a generalisation of Lemma 1.9, which is the degree-one part (one can check that $\mathcal{L}_1^K(H) = (H^{\mathrm{ab}})_K$).

We can devise an analogue of Lemma 2.6 in this context, which gives a criterion for the relative lower central series to stop:

Lemma B.2 *Let a group K act on a group H. Let the set S_K generate K^{ab} and let the set S_H generate $(H^{\mathrm{ab}})_K$. Suppose that for each pair $(s, t) \in S_H^2$ (resp. $(s, t) \in S_H \times S_K$), we can find representatives $\tilde{s}, \tilde{t} \in H$ (resp. $\tilde{s} \in H$ and $\tilde{t} \in K$) of s and t such that $\tilde{s}$ and $\tilde{t}$ commute in H (resp. in $H \rtimes K$). Then $\Gamma_2^K(H) = \Gamma_3^K(H)$, which means that $\Gamma_*^K(H)$ stops at $\Gamma_2^K(H)$, and:*
$$\mathcal{L}(H \rtimes K) \cong (H^{\mathrm{ab}})_K \times \mathcal{L}(K),$$

where the first factor is concentrated in degree one.

Proof. On the one hand, by definition of the relative lower central series, an element of $\mathcal{L}_2^K(H)$ is a sum of brackets in $\mathcal{L}(G \rtimes K)$, either of two elements of $\mathcal{L}_1^K(H)$, or of an element of $\mathcal{L}_1(K)$ with an element of $\mathcal{L}_1(H)$. On the other hand, the relation $[\tilde{s}, \tilde{t}] = 1$ in $H \rtimes K$ readily implies that $[s, t] = 0$ in $\mathcal{L}(H \rtimes K) \cong \mathcal{L}^K(H) \rtimes \mathcal{L}(K)$. Since S_H linearly generates $\mathcal{L}_1^K(H) = (H^{\mathrm{ab}})_K$ and S_K linearly generates $\mathcal{L}_1(K) = K^{\mathrm{ab}}$, we infer that under our hypothesis, all elements of $\mathcal{L}_2^K(H)$ are trivial, which means that $\Gamma_2^K(H) = \Gamma_3^K(H)$. Moreover, from the definition of the relative lower central series $\Gamma_*^K(H)$, this obviously implies that $\Gamma_i^K(H) = \Gamma_{i+1}^K(H)$ for all $i \geqslant 2$. The statement about Lie rings is then just a reformulation of the decomposition $\mathcal{L}(H \rtimes K) \cong \mathcal{L}^K(H) \rtimes \mathcal{L}(K)$ taking into account these conclusions. $\qquad \square$

B.2 Semi-direct products of abelian groups

Let a group G act on an abelian group A. Then the lower central series of $A \rtimes G$ can be computed using linear algebra. Indeed, we have:

$$\Gamma_{k+1}^G(A) := \left[A \rtimes G, \ \Gamma_k^G(A) \right] = \left[G, \ \Gamma_k^G(A) \right].$$

These are commutators in $A \rtimes G$, given, for $g \in G$ and $a \in A$, by:

$$[g, a] = g \cdot a - a = (g - id)(a).$$

As a consequence, $\Gamma_{k+1}^G(A)$ is the subgroup of A generated by the $(g - id)(\Gamma_k^G(A))$ (for $g \in G$), which can be computed by studying the endomorphisms $g - id$ of A.

We now study several instances of this situation. We begin by computing the lower central series of the Klein group $\mathbb{Z} \rtimes \mathbb{Z}$ (which is the fundamental group of the Klein bottle). We then generalise this calculation to any semi-direct product of an abelian group by $\mathbb{Z}$ acting by $-id$. This can be generalised again, to the case of an action of $\mathbb{Z}$ by an involution. Finally, we compute the lower central series of $\Lambda \rtimes W_2$, where W_2 is the Coxeter group of type B_2 (or C_2), acting on the lattice Λ generated by its root system.

B.2.1 The Klein group

There are two distinct automorphisms of $\mathbb{Z}$ (that is, $\pm id$), whence only one non-trivial action of $\mathbb{Z}$ on $\mathbb{Z}$. Thus the following definition makes sense:

Definition B.3 The *Klein group* is the semi-direct product $K = \mathbb{Z} \rtimes \mathbb{Z}$.

Let us denote by x (resp. by t) the element $(1, 0)$ (resp. $(0, 1)$) of $\mathbb{Z} \rtimes \mathbb{Z}$. A presentation of K is given by $K = \langle x, t \mid txt^{-1} = x^{-1} \rangle$.

The lower central series of K decomposes as $\Gamma_*(K) = \Gamma_*^{\mathbb{Z}}(\mathbb{Z}) \rtimes \Gamma_*(\mathbb{Z})$. Thus, in order to understand the lower central series of K, we need to understand the filtration $\Gamma_*^{\mathbb{Z}}(\mathbb{Z})$.

Proposition B.4 *The lower central series of the Klein group is $\Gamma_i(\mathbb{Z} \rtimes \mathbb{Z}) = (2^{i-1}\mathbb{Z}) \rtimes \{1\}$ for $i \geqslant 2$. In other words, $\Gamma_i^{\mathbb{Z}}(\mathbb{Z}) = 2^{i-1}\mathbb{Z}$. In particular, $\mathbb{Z} \rtimes \mathbb{Z}$ is residually nilpotent.*

Proof. This follows from the formula $[x^{2^j}, t] = x^{2^j}(tx^{-2^j}t^{-1}) = x^{2^{j+1}}$, by induction on i. $\qquad \square$

Corollary B.5 *The (graded) Lie ring of the Klein group identifies with $(\mathbb{Z}/2)[X] \rtimes \mathbb{Z}$, where the polynomial ring $(\mathbb{Z}/2)[X]$ is seen as an abelian Lie ring (where X^i is of degree i), and the generator T of $\mathbb{Z}$ (of degree 1) acts via $[X^i, T] = X^{i+1}$.*

Proof. From Proposition B.4, we get a decomposition $\mathcal{L}(K) = \mathcal{L}(2^{*-1}\mathbb{Z}) \rtimes \mathcal{L}(\mathbb{Z})$. Since $\mathbb{Z}$ is abelian, the two factors are abelian Lie rings. The result follows, by calling X^i the class of $x^{2^{i-1}}$ and T the class of t. The formula for brackets comes from $[x^{2^j}, t] = x^{2^{j+1}}$. $\qquad \square$

B.2.2 Generalised Klein groups

Let A be any abelian group, and let $\mathbb{Z}$ act on A via the powers of $-id_A$. The corresponding semi-direct product $K_A = A \rtimes \mathbb{Z}$ is a generalisation of the Klein group $K = K_{\mathbb{Z}}$. We can generalise the above results to this context:

Proposition B.6 *The lower central series of K_A is given by $\Gamma_i(A \rtimes \mathbb{Z}) = (2^{i-1}A) \rtimes \{1\}$ for $i \geqslant 2$. In other words, for all $i \geqslant 2$, $\Gamma_i^{\mathbb{Z}}(A) = 2^{i-1}A$. In particular, for any free abelian group A, K_A is residually nilpotent.*

Proof. Let t denote the generator of $\mathbb{Z}$. For all a in A, we have $[a,t] = a - t \cdot a = 2a$ in $A \rtimes \mathbb{Z}$. Hence $[2^j A, t] = 2^{j+1}A$, from which the calculation of the lower central series follows. Then K_A is residually nilpotent if and only if the intersection of the $2^j A$ is trivial, which is true for instance when A is finitely generated or when A is free. $\qquad\square$

Corollary B.7 *Let us consider the abelian Lie algebra $\mathcal{L}(2^{*-1}A) = \bigoplus 2^{i-1}A/2^i A$ (where the sum is taken over $i \geqslant 1$). The (graded) Lie ring of K_A identifies with $\mathcal{L}(2^* A) \rtimes \mathbb{Z}$, where the generator T of $\mathbb{Z}$ acts via the degree-one map induced by $a \mapsto 2a$.*

Proof. From Proposition B.6, we get a decomposition $\mathcal{L}(K) = \mathcal{L}(2^{*-1}A) \rtimes \mathcal{L}(\mathbb{Z})$. Since $\mathbb{Z}$ and A are abelian, the two factors are abelian Lie rings. The result follows, since brackets with T come from commutators with t, given by $[a,t] = 2a$ in $A \rtimes \mathbb{Z}$. $\qquad\square$

Since t^2 acts trivially on A, it is a central element of K_A (in fact, one easily sees that it generates the center of K_A if A is not trivial). Thus we can consider $A \rtimes (\mathbb{Z}/2)$ (where $\mathbb{Z}/2$ acts on A via $-id_A$) as a quotient of K_A, which behaves much in the same way:

Corollary B.8 *Consider the group $A \rtimes (\mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts on the abelian group A via $-id_A$. We have $\Gamma_i(A \rtimes (\mathbb{Z}/2)) = (2^{i-1}A) \rtimes \{1\}$ for all $i \geqslant 2$. In particular, for any finitely generated abelian group A, $A \rtimes (\mathbb{Z}/2)$ is residually nilpotent. Moreover, the (graded) Lie ring of this group identifies with $\mathcal{L}(2^{*-1}A) \rtimes (\mathbb{Z}/2)$, where the generator T of $\mathbb{Z}/2$ acts via the degree-one map induced by $a \mapsto 2a$.*

Finally, let us spell out the particular case where $A \cong \mathbb{Z}^n$ is free abelian on some basis $x_1, \ldots, x_n$. We then denote K_A by K_n, for short.

Corollary B.9 *The (graded) Lie ring of K_n identifies with $(\mathbb{Z}/2)^n[X] \rtimes \mathbb{Z}$, where the polynomial ring $(\mathbb{Z}/2)^n[X]$ is seen as an abelian Lie ring (where X^i is of degree i), and the generator T of $\mathbb{Z}$ (of degree 1) acts via $[v \cdot X^i, T] = v \cdot X^{i+1}$, for $v \in (\mathbb{Z}/2)^n$.*

Proof. This is just a matter of identifying $\mathcal{L}(2^{*-1}\mathbb{Z}^n)$ with $(\mathbb{Z}/2)^n[X]$, by calling $v \cdot X^i$ the class of the sum of the $2^{i-1}x_k$ such that $v_k = 1$. $\qquad\square$

B.2.3 A further generalisation

Let A be an abelian group, and let $\mathbb{Z}$ act on A via the powers of some involution τ. Let us denote by K_τ the corresponding semi-direct product $A \rtimes \mathbb{Z}$. We have:

$$((\tau - 1) + 2)(\tau - 1) = \tau^2 - 1 = 0,$$

which means that $\tau - 1$ acts via multiplication by -2 on $V := \mathrm{Im}(\tau - 1)$.

Proposition B.10 *The lower central series of K_τ is given by $\Gamma_i(A \rtimes \mathbb{Z}) = (2^{i-2}V) \rtimes \{1\}$ for $i \geqslant 2$. In other words, for all $i \geqslant 2$, $\Gamma_i^{\mathbb{Z}}(A) = 2^{i-2}V$. In particular, for any free abelian group A, K_τ is residually nilpotent.*

Proof. For all $a \in A$, we have $[a,t] = a - \tau(a) = (1-\tau)(a)$. This implies that $\Gamma_2^{\mathbb{Z}}(A) = \mathrm{Im}(\tau - 1) = V$. Then for all v in V, we have $[v,t] = v - \tau(v) = 2v$ in $A \rtimes \mathbb{Z}$, and the rest of the proof is similar to the proof of Proposition B.6. $\qquad\square$

Corollary B.7 generalises immediately to this context:

Corollary B.11 *The (graded) Lie ring of K_τ identifies with $(A/V \oplus V/2V \oplus 2V/4V \oplus \cdots) \rtimes \mathbb{Z}$, where A/V and $\mathbb{Z}$ are in degree 1, $2^{i-2}V/2^{i-1}V$ is in degree i and the generator T of $\mathbb{Z}$ acts via the degree-one map induced by $1 - \tau$ (which coincides with $v \mapsto 2v$ on V).*

The reader can also easily write a generalisation of Corollary B.8 to this context, by factoring the action of $\mathbb{Z}$ through $\mathbb{Z}/2$.

B.2.4 More actions on abelian groups

Let us consider the group $\langle \sigma, \gamma \mid \sigma^2 = \gamma^2 = (\sigma\gamma)^4 = 1 \rangle$, which is the Coxeter group of type B_2, also denoted by $W_2 = (\mathbb{Z}/2) \wr \mathfrak{S}_2$ in the rest of the paper. It acts on $\mathbb{R}^2$ in the usual way: γ acts by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and σ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This action preserves the lattice $\Lambda := \mathbb{Z} \cdot (0,1) \oplus \mathbb{Z} \cdot (\frac{1}{2}, \frac{1}{2})$ (which is generated by roots). It also preserves the lattice $V = \mathbb{Z}^2$, which is of index 2 in Λ.

Proposition B.12 *The filtration $\Gamma_*^{W_2}(\Lambda)$ on Λ is given by:*

$$\Lambda \supset V \supset 2\Lambda \supset 2V \supset 4\Lambda \supset \cdots .$$

In particular, $\Lambda \rtimes W_2$ is residually nilpotent, but not nilpotent.

Proof. One can easily write down explicitly the eight matrices for the actions of elements of W_2 (which are the invertible monomial matrices in $\mathrm{GL}_2(\mathbb{Z})$). Recall that for every $g \in W_2$, $g - id$ is the commutator by g in $\Lambda \rtimes W_2$. It is then easy to check that the $g - id$ send Λ to V (resp. V to 2Λ), and that the $(g - id)(\Lambda)$ (resp. the $(g - id)(V)$) generate V (resp. 2Λ), whence the result. $\square$

Remark B.13 One can look at the proof in a geometric way, by seeing each element $g \cdot v - v$ as the difference between two vertices of a square centered at 0.

Remark B.14 One can compute completely the associated Lie ring, which is a semi-direct product of the abelian Lie ring $(\mathbb{Z}/2)[X]$ by the mod 2 Heisenberg Lie ring $\mathcal{L}(W_2) \cong \mathfrak{n}_3(\mathbb{Z}/2)$.

Remark B.15 Let us use the notations $a := (0,1)$, $b := (1,0)$ and $c := (\frac{1}{2}, \frac{1}{2})$. Then Λ can be described abstractly as the abelian group generated by a, b and c, modulo the relation $a + b = 2c$. Moreover, the action of σ fixes c and exchanges a and b, and the action of γ is via $a \mapsto a$, $b \mapsto -b$ and $c \mapsto c - b$. In particular, since $b = 2c - a$ and $c - b = a - c$, in the basis (a, c) of Λ, σ acts by $\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$ and γ by $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Notice that in this basis, V is just the subgroup of elements whose second coordinate is even.

In the course of the proof of Proposition 6.68, we encounter a slight variation on the above action of W_2 on $\Lambda \cong \mathbb{Z}^2$. Namely, we can construct an action of $K = Dic_{16} \times (\mathbb{Z}/2)$ on Λ, by making $Dic_{16} = \langle \sigma, \gamma \rangle$ act through its quotient W_2 (which is the quotient by σ^2), and making $\mathbb{Z}/2 = \langle \tau \rangle$ act by $-id$. Notice that there is already an element in W_2 acting by $-id$, namely the central element $(\sigma\gamma)^2$ of W_2, so this action is in fact through the quotient $K \twoheadrightarrow W_2$ sending respectively σ, γ and τ to σ, γ and $(\sigma\gamma)^2$. In particular, this implies that we have:

$$\Gamma_*^K(\Lambda) = \Gamma_*^{W_2}(\Lambda).$$

B.3 Free products

B.3.1 Two examples

Consider the simplest free product of two groups, which is the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2 = \langle x, y \mid x^2 = y^2 = 1 \rangle$. We can determine its lower central series from its description as a semi-direct product:

Proposition B.16 *There is an isomorphism:*

$$\mathbb{Z}/2 * \mathbb{Z}/2 \;\cong\; \mathbb{Z} \rtimes (\mathbb{Z}/2).$$

As a consequence, this group is residually nilpotent, and its Lie ring is $(\mathbb{Z}/2)[X] \rtimes (\mathbb{Z}/2)$, where both factors are abelian Lie rings, and the generator T of $\mathbb{Z}/2$ acts by $[X^i, T] = X^{i+1}$.

Proof. We know a presentation of each these groups, namely $\mathbb{Z}/2 * \mathbb{Z}/2 = \langle x, y \mid x^2 = y^2 = 1 \rangle$ and $\mathbb{Z} \rtimes (\mathbb{Z}/2) = \langle a, t \mid tat^{-1} = a^{-1},\ t^2 = 1 \rangle$. It is then easy to check that the assignments $x \mapsto t$, $y \mapsto ta$ and $t \mapsto x$, $a \mapsto xy$ define morphisms inverse to each other. The rest is an application of Corollary B.8 with $A = \mathbb{Z}$. $\qquad\square$

Consider now the free product $\mathbb{Z} * (\mathbb{Z}/2) = \langle x, y \mid y^2 = 1 \rangle$. We have a similar decomposition into a semi-direct product:

Proposition B.17 *There is an isomorphism:*

$$\mathbb{Z} * (\mathbb{Z}/2) \;\cong\; \mathbf{F}_2 \rtimes (\mathbb{Z}/2),$$

where the action of the generator t of $\mathbb{Z}/2$ is given by exchanging the two elements a and b of a basis of the free group $\mathbf{F}_2$.

Proof. Again, we know a presentation of each these groups, namely $\mathbb{Z} * (\mathbb{Z}/2) = \langle x, y \mid y^2 = 1 \rangle$ and $\mathbf{F}_2 \rtimes (\mathbb{Z}/2) = \langle a, b, t \mid tat^{-1} = b,\ tbt^{-1} = a,\ t^2 = 1 \rangle$. It is then easy to check that the assignments $x \mapsto tb$, $y \mapsto t$ and $t \mapsto y$, $a \mapsto xy$, $b \mapsto yx$ define morphisms inverse to each other. One may alternatively observe that the Tietze transformation removing the generator a turns the second presentation into the first. $\qquad\square$

Remark B.18 The group $\mathbb{Z} * (\mathbb{Z}/2)$ is isomorphic to $\mathbf{wB}_2$ (or $\mathbf{vB}_2$), and the above isomorphism can be identified with $\mathbf{wB}_2 \cong \mathbf{wP}_2 \rtimes \mathfrak{S}_2$, together with $\mathbf{wP}_2 \cong \mathbf{F}_2$ (with basis (χ_{12}, χ_{21})).

The lower central series of $\mathbb{Z} * (\mathbb{Z}/2)$ is much more difficult to compute than the one of $\mathbb{Z}/2 * \mathbb{Z}/2$ above. The reader can find a presentation of the associated Lie ring in [Lab77], where the methods used are somewhat different from ours. Our methods could be adapted to recover this result, together with the residual nilpotence of the group, but we will not do so here. We only give a proof of the following:

Proposition B.19 *The group $\mathbb{Z} * (\mathbb{Z}/2)$ is residually nilpotent, but not nilpotent. Its Lie ring has only 2-torsion elements, except in degree one.*

Proof. Note that the group $\mathbb{Z} * (\mathbb{Z}/2)$ surjects onto $\mathbb{Z}/2 * \mathbb{Z}/2$, whose lower central series does not stop, by Proposition B.16. Notice also that the statement about torsion has already been proved in Example 2.9. As a consequence, we only need to prove that $\mathbb{Z} * (\mathbb{Z}/2)$ is residually nilpotent, which is the difficult part of the statement. We can prove it using some kind of Magnus expansion. Namely, let us consider the associative algebra of non-commutative formal power series $A := \mathbf{F}_2\langle\!\langle X, Y \rangle\!\rangle / (Y^2 = 1)$. We get a morphism Φ from $\mathbb{Z} * (\mathbb{Z}/2)$ to $A^\times$ by sending x to $1 + X$ and y to $1 + Y$. It is injective, by the usual argument: if $x^{a_1} y^{\epsilon_1} \cdots x^{a_l} y^{\epsilon_l} x^{a_{l+1}}$ is a non-trivial reduced expression of some non-trivial element $g \in \mathbb{Z} * (\mathbb{Z}/2)$ (with $\epsilon_i = \pm 1$, $a_i \in \mathbb{Z}$, and a_1 and a_{l+1} possibly trivial), then, by writing $a_i = 2^{b_i}(2c_i + 1)$, using $(1 + X)^{2^k} = 1 + X^{2^k}$ and $(1 + T)^\alpha = 1 + \alpha T + \cdots$, we see that the coefficient of the monomial $X^{2^{b_1}} Y X^{2^{b_2}} Y \cdots Y X^{2^{b_{l+1}}}$ in $\Phi(g)$ is not trivial, hence $\Phi(g) \neq 1$.

Now let us denote by (X, Y) the ideal generated by X and Y in A, and by $A_k^\times$ the subgroup $1 + (X, Y)^k$ of $A^\times$ (for $k \geqslant 1$). It is easy to see that $[A_1^\times, A_k^\times] \subset A_{k+1}^\times$ for all $k \geqslant 1$. As a consequence, for all $k \geqslant 1$, we have $\Gamma_k(A_1^\times) \subseteq A_k^\times$. Since the intersection of the $1 + (X, Y)^k$ is obviously trivial, $A_1^\times$ is residually nilpotent, whence also $\mathbb{Z} * (\mathbb{Z}/2)$, which is isomorphic to one of its subgroups. $\qquad\square$

B.3.2 An Artin group

Consider the Artin group of type B_2 (which is also $\mathbf{B}_{1,2}$ – see Lemma 3.15), that is:

$$G := \langle \sigma, x \mid (\sigma x)^2 = (x\sigma)^2 \rangle.$$

Let $\delta := \sigma x$. Then $(\sigma x)^2 = (x\sigma)^2$ is equivalent to $\delta^2 = \sigma^{-1}\delta^2\sigma$, so that:

$$G = \langle \sigma, \delta \mid \delta^2\sigma = \sigma\delta^2 \rangle.$$

The element δ^2 commutes with the generators δ and σ, hence it is central in G. Since G^{ab} is free on the classes of δ and σ (as is obvious from the presentation), δ^2 is of infinite order. Moreover, the above relation clearly becomes trivial when δ^2 is killed, so that:

$$G/\delta^2 = \langle \sigma, \delta \mid \delta^2 = 1 \rangle \cong \mathbb{Z} * (\mathbb{Z}/2).$$

We thus have a central extension:

$$\mathbb{Z} \lhook\joinrel\longrightarrow G \longrightarrow\!\!\!\!\!\rightarrow \mathbb{Z} * (\mathbb{Z}/2).$$

Proposition B.20 *The group $G = \langle \sigma, x \mid (\sigma x)^2 = (x\sigma)^2 \rangle$ is residually nilpotent (but not nilpotent).*

Proof. We have observed above that the central subgroup $\langle \delta^2 \rangle$ injects into G^{ab}. Thus, we can deduce the (strict) residual nilpotence of G from the fact that $\mathbb{Z} * (\mathbb{Z}/2)$ is (strictly) residually nilpotent (see Proposition B.19) by applying Corollary 1.3. $\square$

The decomposition of G into a central extension can also be used to describe its Lie ring. Namely, it can be obtained from the Lie ring of $\mathbb{Z} * (\mathbb{Z}/2)$ described in [Lab77]; see §B.3.1.

Proposition B.21 *The Lie ring of $G = \langle \sigma, x \mid (\sigma x)^2 = (x\sigma)^2 \rangle$ is a central extension of $\mathcal{L}(\mathbb{Z}*(\mathbb{Z}/2))$ by $\mathbb{Z}$, concentrated in degree one. Precisely, $2\overline{\sigma x}$ is central in $\mathcal{L}(G)$, and $\mathcal{L}(G)/(2\overline{\sigma x}) \cong \mathcal{L}(\mathbb{Z}*(\mathbb{Z}/2))$.*

Proof. We have seen that $\langle \delta^2 \rangle$ injects into G^{ab}, which means that $\langle \delta^2 \rangle \cap \Gamma_2(G) = \{1\}$. As a consequence, the projections $\pi \colon \Gamma_k(G) \twoheadrightarrow \Gamma_k(G/\delta^2)$ have trivial kernels for $k \geqslant 2$, which means that they are isomorphisms. Thus, the canonical morphism from $\mathcal{L}(G)$ to $\mathcal{L}(G/\delta^2)$ is an isomorphism in degree at least 2. In degree one, it identifies with the projection of $G^{\mathrm{ab}} \cong \mathbb{Z}^2$ onto $(G/\delta^2)^{\mathrm{ab}} \cong \mathbb{Z} \times (\mathbb{Z}/2)$, whose kernel is generated by $\overline{\delta^2} = 2\overline{\delta}$. Moreover, since δ^2 is central in G, its class in $\mathcal{L}(G)$ must be central, whence our result. $\square$

B.4 Wreath products

This section is devoted to the study of the lower central series of wreath products. Recall that if K is a group, X a finite set and $\varphi \colon K \to \mathfrak{S}_X$ a morphism, then we can form the *wreath product* of K with any group G:

$$G \wr K := G^X \rtimes K,$$

where K acts through $\varphi \colon K \to \mathfrak{S}_X$ and the action of $\mathfrak{S}_X$ on G^X by permuting the factors. The lower central series $G \wr K$ decomposes as $\Gamma_*^K(G^X) \rtimes \Gamma_*(K)$. But $\Gamma_*^K(G^X) = \Gamma_*^{\varphi(K)}(G^X)$ depends only on the image of K in $\mathfrak{S}_X$. As a consequence, in order to study the lower central series of $G \wr K$ when that of K is known, we can restrict to the case where K is a *subgroup* of $\mathfrak{S}_X$ (and φ is the corresponding inclusion).

Lemma 1.9 allows us to compute abelianisations of wreath products:

Lemma B.22 *Notations being as above, $(G \wr K)^{\mathrm{ab}} \cong (G^{\mathrm{ab}})^{X/K} \times K^{\mathrm{ab}}$. In particular, for any integer $n \geqslant 2$, we have $(G \wr \mathfrak{S}_n)^{\mathrm{ab}} \cong G^{\mathrm{ab}} \times (\mathbb{Z}/2)$.*

Proof. It is a direct consequence of Lemma 1.9:

$$(G \wr K)^{\mathrm{ab}} = ((G^X)^{\mathrm{ab}})_K \times K^{\mathrm{ab}} = ((G^{\mathrm{ab}})^X)_K \times K^{\mathrm{ab}} = (G^{\mathrm{ab}})^{X/K} \times K^{\mathrm{ab}}.$$

In particular, when $K = \mathfrak{S}_n$ with $n \geqslant 2$, then $K^{\mathrm{ab}} = \mathbb{Z}/2$ and K acts transitively on $X = \{1, ..., n\}$, whence the result. $\square$

Corollary B.23 *For any partition $\lambda = (n_1, \ldots, n_l)$ of n, if l' denotes the number of indices $i \leqslant l$ such that $n_i \geqslant 2$, we have $(G \wr \mathfrak{S}_\lambda)^{\mathrm{ab}} \cong (G^{\mathrm{ab}})^l \times (\mathbb{Z}/2)^{l'}$.*

Proof. $G \wr \mathfrak{S}_\lambda$ decomposes as the direct product of the $G \wr \mathfrak{S}_{n_i}$, and $(G \wr \mathfrak{S}_{n_i})^{\mathrm{ab}}$ identifies with G^{ab} when $n_i = 1$, and with $G^{\mathrm{ab}} \times (\mathbb{Z}/2)$ if $n_i \geqslant 2$. $\square$

B.4.1 Partitioned wreath product

We first consider the case where $K = \mathfrak{S}_\lambda$ for some partition $\lambda = (n_1, \ldots, n_l)$ of an integer n, which is the case used throughout the paper. We note that:

$$G \wr \mathfrak{S}_\lambda \cong \prod_{i=1}^{l} G \wr \mathfrak{S}_{n_i}.$$

As a consequence, we only need to consider the case where $K = \mathfrak{S}_n$. We first show that there is a stable behaviour, occurring as soon as $n \geqslant 3$. Then we study the case $n = 2$ (the case $n = 1$ being trivial), which we solve for G abelian.

Recall that the usual generators τ_i of $\mathfrak{S}_n$ are conjugate to each other, hence $\mathfrak{S}_n^{\mathrm{ab}} \cong \mathbb{Z}/2$ is generated by their common class τ, and $\Gamma_2 = \Gamma_\infty$ for $\mathfrak{S}_n$.

Proposition B.24 *Let G be a group. For any $n \geqslant 3$, then the $\tau_i \tau_j^{-1}$ normally generate $\Gamma_2(G \wr \mathfrak{S}_n)$, and $(G \wr \mathfrak{S}_n)^{\mathrm{ab}} \cong G^{\mathrm{ab}} \times \mathbb{Z}/2$. Moreover, the lower central series of $G \wr \mathfrak{S}_n$ stops at Γ_2.*

Proof. Let N be the subgroup of $G \wr \mathfrak{S}_n$ normally generated by the $\tau_i \tau_j^{-1}$. These are in $\Gamma_2(\mathfrak{S}_n)$, whence also in $\Gamma_2(G \wr \mathfrak{S}_n)$, hence the latter contains N. In order to show the converse inclusion, we need to show that $(G \wr \mathfrak{S}_n)/N$ is abelian. For any $g \in G$, let us denote by $\overline{g}$ the class of $(g, 1, ..., 1) \in G^n$ modulo N. We now show that the $\overline{g}$, together with τ, generate $(G \wr \mathfrak{S}_n)/N$, and we use a disjoint support argument to show that they commute with one another.

First, let us remark that $\overline{g}$ commutes with τ. This comes from the fact that τ_2 acts trivially on $(g, 1, ..., 1)$, thus commutes with it in $G \wr \mathfrak{S}_n$. From this, we deduce that $\overline{g}$ is also the class of $\sigma(g, 1, ..., 1)\sigma^{-1} = (1, ..., 1, g, 1, ..., 1)$ for any $\sigma \in \mathfrak{S}_n$ (whose class modulo N is a power of τ). In particular, for all $g, h \in G$, $(g, 1, ..., 1)$ commutes with $(1, h, ..., 1)$, hence $\overline{g}$ commutes with $\overline{h}$.

Now, every $(g_1, ..., g_n) \in G^n$ is the product of the $(1, ..., 1, g_i, 1, ..., 1)$, so that all the elements $(1, ..., 1, g, 1, ..., 1)$ (for all $g \in G$ and any choice of position), together with the τ_i, generate $G \wr \mathfrak{S}_n$. This implies that their classes $\overline{g}$, together with τ, generate $(G \wr \mathfrak{S}_n)/N$. Since these generators commute with one another, this ends the proof that $N = \Gamma_2(G \wr \mathfrak{S}_n)$.

The rest of the statement is a direct application of Lemmas B.22 and B.2. $\square$

Corollary B.25 *Let G be a group, $n \geqslant 3$ be an integer, and $\lambda = (n_1, ..., n_l)$ be a partition of n with $n_i \geqslant 3$ for all i. Then the $\tau_\alpha \tau_\beta^{-1}$ for α and β in the same block of λ normally generate $\Gamma_2(G \wr \mathfrak{S}_\lambda)$, and $(G \wr \mathfrak{S}_\lambda)^{\mathrm{ab}} \cong (G^{\mathrm{ab}} \times \mathbb{Z}/2)^l$. Moreover, the lower central series of $G \wr \mathfrak{S}_\lambda$ stops at Γ_2.*

Proof. Apply Proposition B.24 to each factor of the decomposition $G \wr \mathfrak{S}_\lambda \cong \prod_{i=1}^{l} G \wr \mathfrak{S}_{n_i}$. $\square$

Since $G \wr \mathfrak{S}_1 = G$, the only case left in our study of $G \wr \mathfrak{S}_\lambda$ is the case of $G \wr \mathfrak{S}_2$, which can be quite complicated. We treat the case where G is an abelian group, which we denote by A.

Proposition B.26 *Let us denote by δA the subgroup $\{(a, -a) \mid a \in A\}$ of A^2. For all $i \geqslant 2$, we have $\Gamma_i(A \wr \mathfrak{S}_2) = 2^{i-2}(\delta A)$. Moreover, the Lie algebra decomposes as:*

$$\mathcal{L}(A \wr \mathfrak{S}_2) \cong (A \oplus A/2A \oplus 2A/4A \oplus \cdots) \rtimes (\mathbb{Z}/2),$$

where A and $\mathbb{Z}/2$ are in degree 1 and each factor of the form $2^{i-2}A/2^{i-1}A$ is in degree i. The Lie ring $A \oplus A/2A \oplus \cdots$ is abelian, and the generator T of $\mathbb{Z}/2$ acts on it via the degree-one map:

$$\begin{cases} a \mapsto \overline{a} & \text{in degree } 1, \\ \overline{a} \mapsto \overline{2a} & \text{in degree at least } 2. \end{cases}$$

Proof. This is a straightforward application of Proposition B.10 and Corollary B.11 (adapted to an action of $\mathfrak{S}_2 \cong \mathbb{Z}/2$ instead of $\mathbb{Z}$). Namely, $V = \delta A$ is the subspace of A^2 on which $\mathfrak{S}_2$ acts by $-id$. Moreover, $A^2/V \cong A$ (via $\overline{(a,0)} \leftarrow a$), and the map induced by $1 - \tau$ identifies with the one described in our statement. $\square$

Corollary B.27 *Let G be a group, and λ be a partition with at least one block of size 2. Suppose that $2^i G^{\mathrm{ab}} \neq \{0\}$ for every integer i. Then the lower central series of $G \wr \mathfrak{S}_\lambda$ does not stop.*

Proof. The group $G^{\mathrm{ab}} \wr \mathfrak{S}_\lambda$ is a quotient of $G \wr \mathfrak{S}_\lambda$, whose lower central series does not stop by Proposition B.26. Thus, our statement follows from Lemma 1.1. $\square$

B.4.2 General wreath products

Let us investigate the lower central series of more general wreath products. The following results are not used elsewhere in the paper, but they appear as yet another example of an application of our methods.

We go back to the notations of the beginning of the section: the wreath product $G \wr K$ is taken with respect to an action of K on a finite set X. In the proof of Lemma B.22, one can see that the quotient map $G^X \twoheadrightarrow (G^{\mathrm{ab}})^{X/K}$ is given explicitly by:

$$(g_x)_{x \in X} \mapsto \left(\sum_{x \in C} \overline{g}_x \right)_{C \in X/K}.$$

Suppose then that K acts on X without fixed points. Then every pair of elements in $(G^{\mathrm{ab}})^{X/K}$ can be lifted to a pair of commuting elements in G^X. Indeed, let us choose two distinct elements $x_C, y_C \in C$ in each orbit $C \in X/K$. Then $(\overline{g}_C)_{C \in X/K}$ and $(\overline{h}_C)_{C \in X/K}$ are, respectively, the class of (u_x) and (v_x) where $u_{x_C} = g_C$ (resp. $v_{y_C} = h_C$) and all the other u_x (resp. v_x) are equal to 1. Since (u_x) and (v_x) have disjoint support, they commute.

Under the right (stronger) hypothesis, we can elaborate on this disjoint support argument to get:

Proposition B.28 *Suppose that K acts on X without fixed points. Suppose moreover that we can find a set S generating K such that for all $s \in S$, each K-orbit of X has a fixed point under the action of s. Then for any group G, $\Gamma_*^K(G^X)$ stops at $\Gamma_2^K(G^X)$, and:*

$$\mathcal{L}(G \wr K) \cong (G^{\mathrm{ab}})^{X/K} \times \mathcal{L}(K),$$

where the first factor is concentrated in degree one.

The hypothesis is a stability hypothesis: we suppose that each orbit is large enough, but also large enough with respect to the supports of the generators of K. This is satisfied for instance when K is generated by transpositions (which means exactly that K is conjugate to $\mathfrak{S}_\lambda$ for some partition λ of n), and all its orbits are of size at least 3 (that is, λ has no block of size 1 or 2). Thus, apart from the statements about normal generation, we can recover Proposition B.24 and Corollary B.25 as corollaries of the above proposition.

Proof of Proposition B.28. Recall that $\mathcal{L}_1^K(G^X) \cong ((G^X)^{\mathrm{ab}})_K \cong (G^{\mathrm{ab}})^{X/K}$, so that this result can be obtained as a direct application of Lemma B.2. In fact, we have already explained (just before our proposition) that elements of $(G^{\mathrm{ab}})^{X/K}$ have commuting representatives when the action of K on X is without fixed points. Now, our second hypothesis will allow us to construct, for any $s \in S$, a representative commuting with s of any element of $(G^{\mathrm{ab}})^{X/K}$. We can then apply Lemma B.2 by taking as S_K the image of S in K^{ab}, and as S_H any generating family of $H = G^X$.

Let us fix $s \in S$, and let us choose one fixed point x_C of C for any K-orbit C in X. Let $(\overline{g}_C)_{C \in X/K}$ be an element of $(G^{\mathrm{ab}})^{X/K}$ (where the g_C are elements of G). Define $(u_x) \in G^X$ by $u_{x_C} = g_C$ and $u_x = 1$ when x is not among the x_C. Then (u_x) is a lift of $(\overline{g}_C)$ to G^X, and s acts trivially on it, which means exactly that s commutes with it in $G^X \rtimes K$. $\qquad\square$

Decomposition into orbits. We can enlarge the domain of application of Proposition B.28 by considering the decomposition of X into K-orbits. For instance, if K does have fixed points in X, then if $T \subseteq X$ is the set of such fixed points, we have $G \wr K \cong G^T \times (G \wr K)$, where the second wreath product is with respect to the action of K on $X - T$, which is without fixed points.

In general, when a group K acts on a group H and H decomposes as a direct product $H_1 \times H_2$ of K-groups (that is, a direct product of groups such that the action of K stabilises the factors), then it is clear (from the definition) that $\Gamma_i^K(H) = \Gamma_i^K(H_1) \times \Gamma_i^K(H_2)$ for all $i \geqslant 1$. This implies that $\mathcal{L}^K(H) = \mathcal{L}^K(H_1) \times \mathcal{L}^K(H_2)$, and the Lie algebra $\mathcal{L}(H \rtimes K)$ decomposes as $\left(\mathcal{L}^K(H_1) \times \mathcal{L}^K(H_2)\right) \rtimes \mathcal{L}(K)$, where the action of $\mathcal{L}(K)$ on the product is the diagonal one.

In the case of wreath products, we have a decomposition into a product of K-groups:

$$G^X \cong \prod_{C \in X/K} G^C.$$

As a consequence, $\mathcal{L}^K(G^X)$ identifies with the product of the $\mathcal{L}^K(G^C)$, and this decomposition is preserved by the action of $\mathcal{L}(K)$. Note further that $\Gamma_*^K(G^C)$ does not depend on the whole of K, but only on its image K_C in $\mathfrak{S}_C$. In particular, if we want to apply Proposition B.28 to show that the relative lower central series of G^C stops, we only need to look for generators of K_C, which may be easier to study than generators of K.

Remark B.29 The above decomposition extends to a product decomposition of $G \wr K$ when K identifies with the direct product of its images K_C in the $\mathfrak{S}_C$; then $G \wr K$ is the product of the $G \wr K_C$. This holds in particular for $K = \mathfrak{S}_\lambda$, for which we recover the decomposition of $G \wr \mathfrak{S}_\lambda$ as the product of the $G \wr \mathfrak{S}_{n_i}$ used above.

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