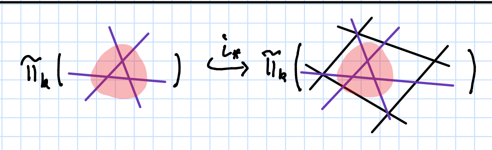
On the $K(\pi,1)$-problem for hyperplane arrangements



Moduli & Friends seminar 9.2.2026

- $\mathcal{A}$ arrangement of hyperplanes in $V = \mathbb{C}^\ell$
- $L = L(\mathcal{A}) = \{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \}$ intersection lattice (poset)

  $X \leq_L Y \;:\Leftrightarrow\; X \supseteq Y$
- $M = M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$ complement

Problem

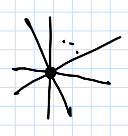
When is M a $K(\pi,1)$-space, i.e.,
when are $\pi_k(M) = 0 \;\; \forall k \geq 2$?

Is this determined by L ?

$\exists$ Zariski-pairs, i.e. Arrs $\mathcal{A}, \mathcal{B}$ with $L(\mathcal{A}) \cong L(\mathcal{B})$
but $\pi_1(M(\mathcal{A})) \not\cong \pi_1(M(\mathcal{B}))$

First examples

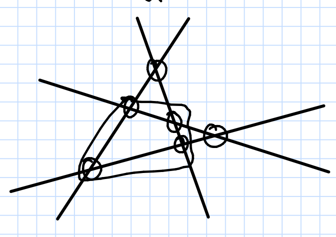
- $\mathrm{rk}(\mathcal{A}) = \mathrm{codim}_V \left(\bigcap_{H \in \mathcal{A}} H \right) = 2$



  are $K(\pi,1)$
  since $M \cong \mathbb{C}^\times \times \mathbb{C} \setminus \{z_1, \dots, z_r\}$
  & $\pi_1 = \mathbb{Z} \times F_r$

- The Boolean arrangement $\mathcal{B}_\ell := \{ \{x_i = 0\} \mid i = 1, \dots, \ell \}$
  is $K(\pi,1)$ since $M \cong (\mathbb{C}^\times)^\ell$
  & $\pi_1 = \mathbb{Z}^\ell$

- generic arrangements, i.e. $n = |\mathcal{A}| > \mathrm{rk}(\mathcal{A}) \geq 3$
  $\forall X \in L \setminus \{\bigcap_{H \in \mathcal{A}} H\}$: $|\mathcal{A}_X = \{ H \in \mathcal{A} \mid H \supseteq X \}| = \mathrm{codim}_V(X)$
  Boolean

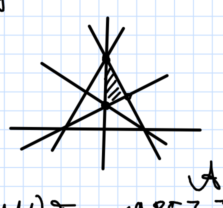


  are not $K(\pi,1)$
  [Hattori's Thm 1975
  $M \simeq_{ht} (T^n / \sim)^{(\mathrm{rk}(\mathcal{A}))}$]

- Braid arrangements $\mathcal{A}(A_\ell) = \{ \{x_i = x_j\} \mid 1 \leq i < j \leq \ell+1 \}$
  are $K(\pi,1)$ [Fadell–Neuwirth 1962:
  $M = \mathrm{Conf}(\mathbb{C}, \ell+1) \longrightarrow \mathrm{Conf}(\mathbb{C}, \ell)$ fibration,
  fiber $= \mathbb{C} \setminus \{z_1, \dots, z_\ell\}$]

  $\pi_1 = PBr_{\ell+1}$ pure Braid group
  $0 \to PBr_{\ell+1} \to Br_{\ell+1} \to W(A_\ell) \to 0$, $W(A_\ell) \cong S_{\ell+1}$

Thm [Deligne 1972]
Simplicial cpx fied real arrangements (i.e. $\mathbb{R}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ consists of simplicial cones)
are $K(\pi,1)$



in particular,
for W real reflection group
$\mathcal{A}(W)$ is $K(\pi,1)$.

Thm [(Falk–Randell) Terao 1985]
If $X \in L$ is modular ($X +_V Y \in L \; \forall Y \in L$) of codim $\mathrm{rk}(\mathcal{A}) - 1$
and $p : V \longrightarrow V/X$, then
$p|_M : M \longrightarrow M(\mathcal{A}_X / X)$ is fibration,
fiber $= \mathbb{C} \setminus \{z_1, \dots, z_r\}$, $r = |\mathcal{A} \setminus \mathcal{A}_X|$
so $\mathcal{A}_X$ is $K(\pi,1)$ $\Rightarrow$ $\mathcal{A}$ $K(\pi,1)$.
$\rightsquigarrow$ Supersolvable arrs ($\exists\; X_1 < \dots < X_\ell \subseteq L$ maximal flag of modular element)
are $K(\pi,1)$.

Thm [Bessis 2015]
If G is a fin. cpx reflection group, then
$\mathcal{A}(G) = \{ \ker(\mathrm{id} - r) \mid r \in G \text{ reflection} \}$ is $K(\pi,1)$

Thm [Amend–Deligne–Röhrle 2020]
Restrictions $\mathcal{A}^X = \{ X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X \}$ $(X \in L)$
of $\mathcal{A} = \mathcal{A}(G(p,q,\ell))$ are $K(\pi,1)$.

$\{x_i = \zeta x_j\}$, $\{x_i = 0\}$ $\quad \zeta \in \mu(p)$ roots of unity

Necessary criteria

Convex subsets

Now: $\mathcal{A}$ affine arrangements in $V = \mathbb{C}^\ell = \mathbb{R}^{2\ell}$

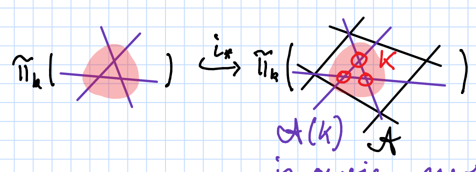
- $K \subseteq V$ convex open
- $\mathcal{A}(K) := \{ H \in \mathcal{A} \mid H \cap K \neq \emptyset \} \subseteq \mathcal{A}$

Lemma
If (*) $L(\mathcal{A}(K)) \overset{\supseteq\checkmark}{=} \{ X \in L(\mathcal{A}) \mid X \cap K \neq \emptyset \}$
then $\pi_k(M(\mathcal{A}(K))) \hookrightarrow \pi_k(M) \;\; \forall k$

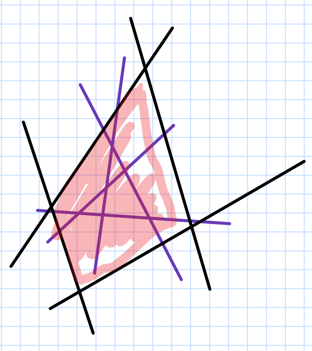
Corollary $\mathcal{A}$ $K(\pi,1)$ $\Rightarrow$ $\mathcal{A}(K)$ $K(\pi,1)$.

Proof: $\exists$ simplicial cpxes Σ & $\Sigma(K)$ with
- $\Sigma \simeq_{ht} M(\mathcal{A})$ & $\Sigma(K) \simeq_{ht} M(\mathcal{A}(K))$
- $\varphi : \Sigma \twoheadrightarrow \Sigma(K)$ simplicial map
  which has a section $\psi : \Sigma(K) \hookrightarrow \Sigma$. $\square$

Recovers: • simple triangle test [Falk–Randell 1986] (rk 3)
for cpx fied real arrs



$\mathcal{A}(K)$ is generic, so not $K(\pi,1)$
$\Rightarrow$ $\mathcal{A}$ not $K(\pi,1)$.



- localization criterion [Oka 1990s]
  $\mathcal{A}$ $K(\pi,1)$ $\Rightarrow$ $\mathcal{A}_X$ $K(\pi,1)$ $\forall X \in L$.
  $\mathcal{A}_X = \mathcal{A}(K)$ for suitable K.

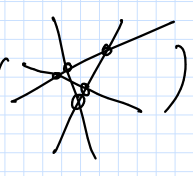
Formality

Now: $\mathcal{A}$ central, i.e., $0 \in \bigcap_{H \in \mathcal{A}} H$

- $\alpha_H \in V^* : \ker(\alpha_H) = H \quad (H \in \mathcal{A})$
- $i_H : \langle \alpha_H \rangle \hookrightarrow V^*$ as $\underbrace{\bigoplus_{H \in \mathcal{A}} \langle \alpha_H \rangle}_{\cong \mathbb{C}^{|\mathcal{A}|}} \xrightarrow{p_1 = \sum_{H \in \mathcal{A}} i_H} V^*$
- $F(\mathcal{A}) := \ker(p_1) = \ker(\alpha_{H_1} | \cdots | \alpha_{H_n}) \in \mathrm{Mat}(\ell \times n)$

Note: For $X \in L$: $j_X : F(\mathcal{A}_X) \hookrightarrow F(\mathcal{A})$

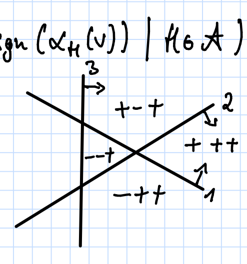
Def [Falk–Randell]
$\mathcal{A}$ is formal $:\Leftrightarrow$ $p_2 = \sum_{X \in L_2,\, \mathrm{codim}(X) = 2} j_X : \bigoplus_{X \in L_2} F(\mathcal{A}_X) \longrightarrow F(\mathcal{A})$ is surjective.

Conf() $\rightarrow$ $F(\mathcal{A}_X) = 0 \;\; \forall X \in L_2$
but $\dim F(\mathcal{A}) = \dim \ker(3 \times 4) = |\mathcal{A}| - \mathrm{rk}(\mathcal{A})$

Thm [Falk–Randell 1986]
$K(\pi,1)$ $\Rightarrow$ formal.

Yoshinaga's consistency

- $\mathcal{A}$ cpx fied real ($\alpha_H \in (\mathbb{R}^\ell)^* \subseteq V^*$)
  $\rightsquigarrow$ $\mathcal{A}_{\mathbb{R}}$ arr in $V_{\mathbb{R}} = \mathbb{R}^\ell$
- $\mathcal{T}(\mathcal{A}) := \{ (\mathrm{sgn}(\alpha_H(v)) \mid H \in \mathcal{A}) \mid v \in V_{\mathbb{R}} \setminus \bigcup_{H \in \mathcal{A}_{\mathbb{R}}} H \} \subseteq \{+,-\}^{\mathcal{A}}$



- for $\sigma \in \{+,-\}^{\mathcal{A}}$, $X \in L$: $\sigma_X = \sigma|_{\mathcal{A}_X} \in \{+,-\}^{\mathcal{A}_X}$

Def [Yoshinaga 2024]
$\mathcal{A}$ is consistent $:\Leftrightarrow$ $\mathcal{T}_2(\mathcal{A}) := \{ \sigma \in \{+,-\}^{\mathcal{A}} \mid \sigma_X \in \mathcal{T}(\mathcal{A}_X) \; \forall X \in L_2 \}$
$= \mathcal{T}(\mathcal{A})$
$\supsetneq$ in general
for simple Δ.

Thm [Yoshinaga 2024]
$K(\pi,1)$ $\Rightarrow$ consistent.

Consistency & formality

$\mathcal{A}$ is formal $:\Leftrightarrow$ $p_2 : \bigoplus_{X \in L_2} F(\mathcal{A}_X) \overset{p_2}{\twoheadrightarrow} F(\mathcal{A}) \overset{j}{\hookrightarrow} \underbrace{\bigoplus_{H \in \mathcal{A}} \langle \alpha_H \rangle}_{=: E} \xrightarrow{p_1} V^*$

- $\hat{V} := \mathrm{Im}(j \circ p_2)^\perp \subseteq E$
- $V = j(F(\mathcal{A}))^\perp$ $\quad (\bigcap_{H \in \mathcal{A}} H = \{0\})$, note: $\mathcal{A} = \{ V \cap E_H \}$, $E_H = \bigoplus_{H' \in \mathcal{A} \setminus \{H\}} \langle \alpha_{H'} \rangle \subseteq E$ hyperplane
- $V_X = j_X(F(\mathcal{A}_X))^\perp \rightsquigarrow \hat{V} = \bigcap_{X \in L_2} V_X$, $\forall X \in L$
  $V_X \supseteq \hat{V} \supseteq V$
- $\sigma \in \{+,-\}^{\mathcal{A}}$: $E^\sigma := \bigcap_{H \in \mathcal{A}} E_H^{\sigma_H}$ intersection of halfspaces in E

$\mathcal{A}$ formal $\Leftrightarrow$ $V = \hat{V}$ $\qquad$ not formal : $V_X \supseteq \hat{V} \supsetneq V$

$\mathcal{T}_2 = \{ \sigma \mid E^\sigma \cap V_X \neq \emptyset \;\; \forall X \in L_2 \}$
$\supsetneq \{ \sigma \mid E^\sigma \cap \hat{V} \neq \emptyset \} \supseteq \{ \sigma \mid E^\sigma \cap V \neq \emptyset \} = \mathcal{T}$
$\Rightarrow$ not consistent

Thm consistent $\Rightarrow$ formal.

Ex: $\{ \{ \sum_{i \in S} x_i = 0 \} \mid S \in 2^{\{1,\dots,4\}} \setminus \{\emptyset\} \}$
in $\mathbb{C}^4$ not known to be $K(\pi,1)$ or not!