

Exotic Spheres lecture course — brief introduction

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An “exotic manifold” – more precisely, an “exotic M ”, where M is your favourite manifold – means a smooth manifold that is homeomorphic to M but not diffeomorphic to M . It is not at all obvious that such a thing really exists, equivalently, that there exists a topological manifold admitting two non-diffeomorphic smooth structures. The first examples of exotic manifolds were discovered by John Milnor in 1956, when he constructed a smooth manifold which is homeomorphic, but not diffeomorphic, to the standard 7-sphere. It is now known that there are precisely 27 distinct (oriented) exotic 7-spheres.

Having discovered the existence of exotic spheres, one would naturally like to classify them. (There are exotic manifolds that are not spheres, such as 4-dimensional Euclidean space, but we will focus on spheres.) It is actually more convenient to consider oriented exotic spheres, since the collection of all of these in a fixed dimension (up to orientation-preserving diffeomorphism), together with the standard sphere, forms a commutative monoid under the operation of connected sum. A complete classification would therefore consist in calculating the precise structure of this monoid and constructing representatives of a set of generators for it. It turns out that – except possibly in dimension 4 where nothing is known – this monoid is an abelian group. Kervaire and Milnor then proved in their paper of 1963 that these abelian groups are all finite (except possibly in dimensions 3 or 4), and showed in principle how to compute them in terms of the stable homotopy groups of spheres (and a little extra information, such as a solution of the Kervaire invariant problem). By now these groups are known explicitly at least up to dimension 64 (except dimension 4). [Note: dimension 3 was not covered by Kervaire and Milnor’s theorem, but it now follows from Perelman’s proof of the 3-dimensional Poincaré Conjecture that this group is trivial.] In particular we know, for example, that there are 27 exotic spheres in dimension 7 and 142211872163171481167115958878207 exotic spheres in dimension 63, but only five exotic spheres in dimension 36. [Note: if we go back to ignoring orientations, these numbers become 14 and 71105936081585740583557979439111 and three respectively.]

In the course we will first set the scene by looking at the relation between exotic spheres, the Poincaré Conjecture and the h -cobordism theorem, and show that the set of homotopy spheres up to h -cobordism forms an abelian group. We will then study different methods of constructing exotic spheres, including Milnor’s original construction via sphere bundles and Brieskorn’s construction as the link of a singularity of a complex variety (Brieskorn-Pham manifolds). The main goal of the course will then be to give a detailed proof of Kervaire and Milnor’s theorem that there are only finitely many exotic spheres in each (high) dimension (where “high” means at least 5), as well as doing some explicit calculations along the way, and seeing how the spheres that we learned to construct earlier fit into the classification. The main tool in their proof is surgery theory (which was in fact invented for this purpose), together with some methods and results from homotopy theory, and provides a beautiful example of the power of surgery techniques in action. Depending on how many lectures this takes so far, we will then study some geometric properties of exotic spheres, such as their curvature.

Prerequisites. This will depend partly on the audience. I will try to make the course as self-contained as possible, and in particular I will not assume knowledge of surgery theory; instead, the necessary methods of surgery theory will be introduced as they are needed in the proof (much like in Kervaire and Milnor’s original paper, where they introduced surgery theory exactly because they needed it for their proof).

Some things that will be assumed:

- Basics of differentiable manifolds, fibre bundles.
- Some algebraic topology: e.g. the Seifert-van-Kampen theorem, Mayer-Vietoris sequence, excision, the Hurewicz theorem, Whitehead’s theorem, universal coefficient theorem, Poincaré duality... (if you have taken the course Topologie I but not yet Topologie II, then [Hatcher, Algebraic Topology] is a good place to learn about the second half of this list).

Useful but optional (these will be used, but they will also be reviewed before we use them):

- Bott periodicity, the J -homomorphism.
- Obstruction theory.
- The Hirzebruch signature theorem.